Lecture 6

(Based on lectures of Pro.Prat)

The Revenue Equivalence Theorem

Direct selling mechanism:

- Probability assignment functions: chance that i gets the object given a vector of reported values

$$p_1(v_1, .., v_N), ..., p_N(v_1, .., v_N),$$

such that $\sum_i p_i \leq 1$ (the auctioneer could keep the object).

- Cost functions: cost paid by bidder i given a vector of reported values (he may pay even if he does not get the object):

$$c_1(v_1, .., v_N), ..., c_N(v_1, .., v_N).$$

The cost could be negative, ie the auctioneer pays the bidder.

The equilibria we have considered in the four formats have corresponding equilibria in direct selling mechanisms.

1. First-Price, Dutch:

$$p_i(v_1, .., v_N) = \left\{ \begin{array}{ll} 1 & \text{if } v_i > v_j \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{array} \right\}$$

$$c_i(v_1, .., v_N) = \left\{ \begin{array}{ll} \hat{b}(v_i) & \text{if } v_i > v_j \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{array} \right\}$$

   Easy to check: truth telling is an equilibrium of this direct mechanism.

2. Second-Price, English: Same assignment function as First Price and

$$c_i(v_1, .., v_N) = \left\{ \begin{array}{ll} \hat{b}(v_{\text{second}}) & \text{if } v_i > v_j \text{ for all } j \neq i \\ 0 & \text{otherwise} \end{array} \right\}$$

For $r_i \in [0,1]$, define:

$$\begin{aligned} \bar{p}_i(r_i) &= \int_0^1 .. \int_0^1 p_i(r_i, v_{-i}) f_{-i}(v_{-i}) dv_{-i}; \\ \bar{c}_i(r_i) &= \int_0^1 .. \int_0^1 c_i(r_i, v_{-i}) f_{-i}(v_{-i}) dv_{-i}. \end{aligned}$$

Then,

$$u_i(r_i, v_i) = \bar{p}_i(r_i) v_i - \bar{c}_i(r_i).$$

A direct mechanism is incentive compatible if

$$u_i(v_i, v_i) \geq u_i(r_i, v_i) \quad \forall i \forall r_i \forall v_i.$$

Proposition 8 *A direct mechanism is incentive-compatible if and only if*

1. $\bar{p}_i(v_i)$ *is non-decreasing in* v_i.

2. $\bar{c}_i$ *can be written as*

$$\bar{c}_i(v_i) = \bar{c}_i(0) + \bar{p}_i(v_i) v_i - \int_0^{v_i} \bar{p}_i(x) dx.$$

Sketch of Proof. For (1), take $v' < v''$ and suppose that $\bar{p}_i(v') > \bar{p}_i(v'')$. Incentive compatibility implies:

$$\begin{aligned} u_i\left(v', v'\right) &\geq u_i\left(v'', v'\right); \\ u_i\left(v'', v''\right) &\geq u_i\left(v', v''\right). \end{aligned}$$

Sum the two inequalities

$$u_i\left(v', v'\right) + u_i\left(v'', v''\right) \geq u_i\left(v'', v'\right) + u_i\left(v', v''\right)$$

That is

$$\bar{p}_i\left(v'\right) v' + \bar{p}_i\left(v''\right) v'' \geq \bar{p}_i\left(v'\right) v'' + \bar{p}_i\left(v''\right) v'.$$

which re-writes as

$$\left(\bar{p}_i\left(v''\right) - \bar{p}_i\left(v'\right)\right)\left(v'' - v'\right) \geq 0,$$

which is a contradiction because $v' < v''$ and $\bar{p}_i(v') > \bar{p}_i(v'')$.

For (2), note that incentive compatibility implies the first-order condition

$$\left.\frac{d}{dr} u_i(r, v)\right|_{r=v} = 0 \quad \forall v.$$

We have

$$\frac{d}{dr}u_i(r,v) = \bar{p}_i'(r)\,v - \bar{c}_i'(r)$$

Hence

$$\bar{c}_i'(v) = \bar{p}_i'(v)\,v \quad \forall v.$$

Integrating both sides

$$\bar{c}_i(v) - \bar{c}_i(0) = \overbrace{\bar{p}_i(v)\,v}^{\text{expected benefit}} - \overbrace{\int_0^v \bar{p}_i(x)\,dx}^{\text{rent}},$$

which corresponds to (2). ■

To interpret (2), go back to mechanism design and think of downward local IC constraints:

$$\hat{t}_i = \hat{t}_{i-1} + u_1(\hat{x}_i,\theta_i) - u_1(\hat{x}_{i-1},\theta_i) \quad \text{for } i = 1,\ldots,n$$

Hence

$$\begin{aligned}
\hat{t}_i &= u_1(\hat{x}_i,\theta_i) - u_1(\hat{x}_{i-1},\theta_i) \\
&\quad + u_1(\hat{x}_{i-1},\theta_{i-1}) - u_1(\hat{x}_{i-2},\theta_{i-1}) \\
&\quad + \ldots \\
&= u_1(\hat{x}_i,\theta_i) \\
&\quad - (u_1(\hat{x}_{i-1},\theta_i) - u_1(\hat{x}_{i-1},\theta_{i-1})) \\
&\quad - \ldots \\
&= u_1(\hat{x}_i,\theta_i) - \sum_{k=1}^{i-1}(u_1(\hat{x}_{i-k},\theta_{i-k+1}) - u_1(\hat{x}_{i-k},\theta_{i-k}
\end{aligned}$$

Theorem 9 (Revenue Equivalence) *If two incentive-compatible direct selling mechanisms have the same probability assignment functions and every bidder with valuation zero is indifferent between the two mechanisms, then the two mechanisms generate the same expected revenue.*

Proof. As the v's are independent, the expected revenue can be written as

$$
\begin{aligned}
& R \\
= \; & \sum_{i=1}^{N} \int_0^1 \bar{c}_i(v_i) f_i(v_i)\, dv_i \\
= \; & \sum_{i=1}^{N} \int_0^1 \left(\bar{c}_i(0) + \bar{p}_i(v_i)\, v_i - \int_0^{v_i} \bar{p}_i(x)\, dx \right) f_i(v_i)\, dv_i \\
= \; & \sum_{i=1}^{N} \bar{c}_i(0) + \sum_{i=1}^{N} \int_0^1 \left(\bar{p}_i(v_i)\, v_i - \int_0^{v_i} \bar{p}_i(x)\, dx \right) f_i(v_i)\, d
\end{aligned}
$$

The revenue depends only on $\bar{c}_i(0)$ and $\bar{p}_i$, not on $\bar{c}_i(v_i)$. ■

We can now add new formats to the list. See exercise.

Efficiency

The four formats have the same p.

They allocate the object to the bidder with the highest valuation.

Efficient.

Trade-off between efficiency and revenue maximation:

- The auctioneer can increase expected revenue by setting a reserve price.

- A reserve price mis-allocates the object with positive probability.

Reserve Price in Second-Price Auction

The auctioneer uses a second-price auction with reserve price $r \geq 0$. If all the bids are below r, the auctioneer keeps the object (for which she has zero utility).

Proposition 10 *The optimal reserve price is strictly greater than zero.*

As before, $h(v)$ is the density function of the second-highest element v^{second} of $\{v_1, ..., v_N\}$.

The expected revenue is

$$\begin{aligned} R_2 &= \int_r^1 \hat{b}(v) h(v) dv + r \Pr\left(v^{\text{second}} < r, v^{\text{first}} > r\right) \\ &= N(N-1) \int_r^1 v (F(v))^{N-2} f(v) (1 - F(v)) dv \\ &\quad + rN (F(r))^{N-1} (1 - F(r)) \end{aligned}$$

Take derivatives:

$$\begin{aligned}\frac{dR_2}{dr} &= -N(N-1)\,r\,(F(r))^{N-2} f(r)\,(1-F(r)) \\ &\quad + N(N-1)\,r\,(F(r))^{N-2} f(r)\,(1-F(r)) \\ &\quad - rN(N-1)\,(F(r))^{N-1} f(r) \\ &\quad + N(F(r))^{N-1}(1-F(r))\end{aligned}$$

and

$$\frac{\left.\frac{dR_2}{dr}\right|_r}{(F(r))^{N-1}} = -rN(N-1)\,f(r) + N(1-F(r))$$

Then

$$\lim_{r\to 0^+} \frac{\left.\frac{dR_2}{dr}\right|_r}{(F(r))^{N-1}} = N > 0$$

For r small enough, $\left.\frac{dR_2}{dr}\right|_r$ is positive and a higher r increases the auctioneer's expected revenue.

Common Values

So far, we have assumed that valuations are indepently distributed.

But think of auctions for

- Oil fields
- New issues of securities
- Spectrum (UMTS)
- Any object which could be re-sold (paintings, cars, etc).

Values are then interdependent.

Let us look at the most extreme case: the value is the same for every player (but still stochastic):

$$v_1 = ... = v_n = v$$

and v has density f and CDF F on $[0, 1]$

Buyer i observes signal y_i with distribution $g(y_i|v)$. Assume that the y's are independent across buyers conditional on v.

Restrict attention to second-price auctions.

Is the equilibrium of the game

$$b_i(y_i) = E[v|y_i]? \tag{4}$$

No. A buyer who bids $E[x|y_i]$ is paying too much on average.

To see this, suppose everybody bids according to the *naive* strategy in (4). If i wins, it means that

$$E[v|y_i] = \max(E[v|y_1], ..., E[v|y_n])$$

equivalent to

$$y_i = \max(y_1, ..., y_n)$$

But then

$$E[v|y_1, ..., y_n] < E[v|y_i]$$

If i had known what the others know he would have bid less. This is the *winner's curse*.

In equilibrium, rational bidders are not subject to the winner's curse because they do not use a naive strategy.

The equilibrium strategy is the *sophisticated* bid function:

$$\tilde{b}_i(y_i) = E\left[v|y_i, y_i = \max_{j \neq i} y_j\right],$$

i.e. a buyer conditions his bid on the event his bid is equal to the second-highest bid.

Are bidders rational? Experimental evidence (Kagel and Levin 1986): both naive and strategic bidding.

Information Provision and Revenue Maximization

Should the auctioneer allow bidders to get more information about the object for sale?

Example: provide an independent expert report.

Suppose the cost of information provision is zero

Milgrom-Weber (Econometrica 1982)

Theorem 11 *In symmetric environments, if the auctioneer uses a first- or second-price auction the best reporting policy is full disclosure.*

In our example:

Suppose the auctioneer chooses between: (1) letting bidders know only y_i; (2) providing them with perfect information (they learn v).

With (2), the bid is simply $b_i = v$ and each buyer gets zero expected payoff.

Which Format?

With common values, the Revenue Equivalence Theorem does not hold.

It is still true that First $=$ Dutch, but

Milgrom-Weber prove:

- English $>$ Second Price.

  Intuition: The sequential format provides more info.

- Second Price $>$ First Price (if bidders are risk-neutral):

  Intuition: reduce winner's curse

Lecture 3: Mechanism Design

Very general setup: several players with private information, one player who can commit.

1. Principal offers mechanism (aka contract or incentive scheme)

2. Each agent accepts/rejects mechanism

3. Agents play according to mechanism

Many stages $\Rightarrow$ not a static Bayesian game

but... Revelation Principle: focus without loss of generality on mechanisms such that:

- all agents accept

- all agents simultaneously and truthfully reveal their types

Main point: incomplete info + possibility to reject $\Rightarrow$ inefficiency

Principal: benevolent government or profit maximizer

$I+1$ players. $i=0$ is the principal.

$\theta = (\theta_1, ..., \theta_I) \in \Theta$

$y \in Y$: allocation (decided by principal)

$y_0 \in Y$: default allocation

$u_i(y, \theta)$: utility of i

Mechanism: M_i: message space of each agent. $y_M : M \to Y$: allocation function.

1. Principal announces (M, y_M).

2. $\forall i$, Agent i accepts/rejects.

3. If everybody accepts, agent i chooses $\mu_i(\theta_i) \in M_i$.

4. The allocation is $y = y_M\left(\{\mu_i(\theta_i)\}_{i \in I}\right)$.

Example: First-Price Auction

Agents: bidders with symmetrically distributed valuations $\{\theta_i\}_{i \in I}$

Principal: auctioneer

Allocation: $Y = \left(i^*, \{t_i\}_{i \in I}\right)$

Default allocation: $y_0 = \left(i^* = 0, \{t_i\}_{i \in I} = 0\right)$

Message space = bids: $M_i = [0, \infty)$

Allocation function:

$$y_i^M(m) = \begin{pmatrix} i^* = \arg\max_i m_i \\ t_i = \begin{cases} 0 & \text{if } i \neq i^* \\ m_i & \text{if } i = i^* \end{cases} \end{pmatrix}$$

Bayesian equilibrium (see previous lecture):

$$\mu_i^*(\theta_i) = \frac{1}{(F(\theta_i))^{N-1}} \int_0^{\theta_i} x d\left((F(x))^{N-1}\right)$$

Revelation Principle

Direct mechanism: $M_i = \Theta_i$. Agent i announces $\hat{\theta}_i$. $\bar{y} : \Theta \to Y$.

Truthtelling: $\hat{\theta}_i = \theta_i$.

Step 1 Given the equilibrium of a mechanism in which some agents reject, there exists a mechanism which has an equivalent equilibrium but in which all agents accept.

Step 2 Given the equilibrium of a mechanism in which all agents accept, there exists a direct mechanism which has an equivalent equilibrium and in which all agents reveal their types truthfully.

Take M, y_M, μ^* as given. Construct $\bar{y} : \Theta \to y$ such that

$$\bar{y}\left(\hat{\theta}\right) = y^M\left(\mu^*\left(\hat{\theta}\right)\right) \quad \forall \hat{\theta} \in \Theta.$$

Then,

$$\begin{aligned}
& E_{\theta_{-i}}\left[u_i\left(\bar{y}\left(\theta\right),\theta_i,\theta_{-i}\right)|\theta_i\right] \\
= \;& E_{\theta_{-i}}\left[u_i\left(y^M\left(\mu^*\left(\theta\right)\right),\theta_i,\theta_{-i}\right)|\theta_i\right] \\
= \;& \max_{\mu_i\in M_i} E_{\theta_{-i}}\left[u_i\left(y^M\left(\mu_1^*\left(\theta_1\right),..,\mu_i,..,\mu_I^*\left(\theta_I\right)\right),\theta_i,\theta_{-i}\right)|\theta_i\right] \\
\geq \;& \max_{\hat{\theta}_i\in\Theta_i} E_{\theta_{-i}}\left[u_i\left(\bar{y}\left(\theta_1,..,\hat{\theta}_i,..,\theta_I\right),\theta_i,\theta_{-i}\right)|\theta_i\right]
\end{aligned}$$

Caveat: the Revelation Principle does not say that the *set* of equilibria of the original mech is equal to the *set* of equilibria of the direct mech.

Dominant strategies...

Example (continued): First-Price Auction

Find the direct mechanism and truthful equilibrium corresponding to the classical first-price auction (an indirect mechanism).

Message space = types = valuations: $M_i = \Theta_i = [0, 1]$

Allocation function: Use:

$$
\begin{aligned}
\bar{y}_i\left(\hat{\theta}\right) &= y_i^M\left(\mu^*\left(\hat{\theta}\right)\right) \\
&= \left(\begin{array}{l} i^* = \text{arg max}_i\, \mu_i^*\left(\hat{\theta}_i\right) \\ t_i = \begin{cases} 0 & \text{if } i \neq i^* \\ \mu_i^*\left(\hat{\theta}_i\right) & \text{if } i = i^* \end{cases} \end{array}\right) \\
&= \left(\begin{array}{l} i^* = \text{arg max}_i\, \frac{1}{\left(F\left(\hat{\theta}_i\right)\right)^{N-1}} \int_0^{\hat{\theta}_i} xd\left(\left(F\left(x\right)\right)^{N-1}\right) \\ t_i = \begin{cases} 0 & \text{if } i \neq i^* \\ \frac{1}{\left(F\left(\hat{\theta}_i\right)\right)^{N-1}} \int_0^{\hat{\theta}_i} xd\left(\left(F\left(x\right)\right)^{N-1}\right) & \text{if } i = i^* \end{cases} \end{array}\right) \\
&= \left(\begin{array}{l} i^* = \text{arg max}_i\, \hat{\theta}_i \\ t_i = \begin{cases} 0 & \text{if } i \neq i^* \\ \frac{1}{\left(F\left(\hat{\theta}_i\right)\right)^{N-1}} \int_0^{\hat{\theta}_i} xd\left(\left(F\left(x\right)\right)^{N-1}\right) & \text{if } i = i^* \end{cases} \end{array}\right)
\end{aligned}
$$

The equilibrium of this direct mechanism is truthful: each player i selects message $\hat{\theta}_i = \theta_i$.

For every vector of types θ, the allocation is the same as in the first-price auction.

Usually, $y = (x, t)$

$x \in X$: decision

$t \in [0, \infty)^I$: transfer

	example	x	θ	t
1	price discrim.	quantity, quality	willingness to pay	price
2	regulation	cost	technology	income
3	income tax	income	ability	tax
4	public good	public decision	preference	contribution
5	auction	winner	willingness to pay	price
6	bargaining	trade	preference	price

1. Mussa-Rosen (1978)

2. Baron-Myerson (1982)

3. Mirrlees (1971)

4. Groves (1973)

5. Vickrey (1961)

6. Myerson-Satterthwaite (1983)

Mechanism Design with One Agent

- Example: price discrimination with two types.

- General case.

Example

Seller: marginal cost c. Sells x to buyer at price t.

$$\begin{aligned} u_0 &= t - cx \\ u_1 &= \theta V(x) - t \quad \text{with } V' > 0, V'' < 0, V(0) = 0 \\ \theta &\in \{\theta_L, \theta_H\} \text{ with } \theta_L < \theta_H \text{ and } \Pr[\theta_H] = p \end{aligned}$$

If the seller knew θ,

$$\begin{aligned} t(x,\theta) &= \theta V(x(\theta)) \\ x(\theta) &= \arg\max_x \theta V(x) - cx \end{aligned}$$

Let $(*) = \left(x_L^*, x_H^*, t_L^* = \theta_L V\left(x_L^*\right), t_H^* = \theta_H V\left(x_H^*\right)\right)$ be the full-info solution.

If the seller does not know θ and she offers $(*)$, the agent lies. When $\theta = \theta_H$,

$$\begin{aligned} \hat{\theta} &= \theta_H \text{ yields } 0 \\ \hat{\theta} &= \theta_L \text{ yields } \theta_H V(x_L^*) - \theta_L V(x_L^*) > 0 \end{aligned}$$

The seller should select (x_L, x_H, t_L, t_H) such that:

An agent with θ_L accepts:

$$\theta_L V(x_L) - t_L \geq 0 \qquad (\text{IR}_L)$$

An agent with θ_H accepts:

$$\theta_H V(x_H) - t_H \geq 0 \qquad (\text{IR}_H)$$

An agent with θ_L reports $\hat{\theta} = \theta_L$:

$$\theta_L V(x_L) - t_L \geq \theta_L V(x_H) - t_H \qquad (\text{IC}_L)$$

An agent with θ_H reports $\hat{\theta} = \theta_H$:

$$\theta_H V(x_H) - t_H \geq \theta_H V(x_L) - t_L \qquad (\text{IC}_H)$$

Step 1 If $x_L > x_H$, constraints cannot be satisfied. Sum IC_L and IC_H:

$$(\theta_H - \theta_L)(V(x_H) - V(x_L)) \geq 0$$

Step 2 IC$_H$ binding $\Rightarrow$ IC$_L$:

$$\begin{aligned} \theta_L \left(V\left(x_H\right) - V\left(x_L\right)\right) &\leq \theta_H \left(V\left(x_H\right) - V\left(x_L\right)\right) \\ &= t_H - t_L \end{aligned}$$

Step 3 IC$_H$ and IR$_L$ $\Rightarrow$ IR$_H$:

$$\begin{aligned} \theta_H V\left(x_H\right) - t_H &\overset{IC_H}{\geq} \theta_H V\left(x_L\right) - t_L \\ \geq \ \theta_L V\left(x_L\right) - t_L &\overset{IR_L}{\geq} 0 \end{aligned}$$

Step 4 In the optimal contract IC$_H$ and IR$_L$ are binding (and IC$_L$ and IR$_H$ are satisfied). If IC$_H$ were not binding, increase t_H. If IR$_L$ were not binding, increase t_L.

Step 5 The optimal contract solves

$$\max_{t,x} (1-p)\left(t_L - c x_L\right) + p\left(t_H - c x_H\right)$$

subject to IC$_H$ and IR$_L$ binding:

$$\begin{aligned} t_L &= \theta_L V\left(x_L\right) \\ t_H &= \theta_L V\left(x_L\right) + \theta_H V\left(x_H\right) - \theta_H V\left(x_L\right) \end{aligned}$$

The problem is separable and rewrites as

$$\max_{x_L}(1-p)\left(\theta_L V\left(x_L\right)-cx_L\right)-p\left(\theta_H-\theta_L\right)V\left(x_L\right.$$
$$+\max_{x_H} p\left(\theta_H V\left(x_H\right)-cx_H\right)$$

Compare with the efficient program:

$$\max_{x_L}(1-p)\left(\theta_L V\left(x_L\right)-cx_L\right)+\max_{x_H} p\left(\theta_H V\left(x_H\right)-cx_H\right)$$

$x_L < x_L^*$ and $x_H = x_H^*$.

Lessons: (1) IC_H and IR_L binding; (2) No distortions at the top; (3) Rationing at the bottom.

Mechanism Design with One Agent: General Case

Let $x \in X \subset \Re$, where X is a finite set. Let $\Theta = \{\theta_1, ..., \theta_n\} \in \Re^n$, with respective probabilities $p_1, ..., p_n$. The principal's utility is $u_0(x, \theta) + t$. The agent's is $u_1(x, \theta) - t$.

Let $t_0 = x_0 = 0$ be the normalized outside option of the agent. By the Revelation Principle, the principal's problem is

$$\max_{t,x} \sum_i p_i \left(u_0(x_i, \theta_i) + t_i\right)$$
subject to

$$\begin{aligned} u_1(x_i, \theta_i) - t_i &\geq u_1\left(x_j, \theta_i\right) - t_j \quad \forall i, j \quad (\mathrm{IC}_{i,j}) \\ t_0 &= x_0 = 0 \end{aligned}$$

(j can be 0)

Lattice Theory

A function $f : \Re^2 \to \Re$ is (strictly) *supermodular* if $\forall y'' > y', \forall z'' > z'$

$$f(y'', z'') - f(y', z'') \geq (>) f(y'', z') - f(y', z').$$

If f is twice differentaible, then it is supermodular if and only if $\frac{\partial^2 f}{\partial x \partial y} \geq 0$ everywhere.

1. Implementability

A vector of quantities $x = (x_1, ..., x_n)$ is *implementable* if there exists a $t = (t_1, ..., t_n)$ such that the IC constraints are satisfied.

We assume

$$u_1 \text{ is strictly supermodular in } x \text{ and } \theta. \qquad \text{(A1)}$$

This assumption means that the marginal willingness to pay for x is increasing in the buyer's type.

Proposition 12 *Under A1, x is implementable if and only it is nondecreasing, that is, $x_0 \leq x_1 \leq \cdots \leq x_n$.*

Proof. Step 1: "Only if" part: Suppose $x_i < x_k$ for some $i > k$. Summing IC$_{i,k}$ and IC$_{k,i}$ yields

$$u_1(x_k, \theta_k) - u_1(x_i, \theta_k) \geq u_1(x_k, \theta_i) - u_1(x_i, \theta_i),$$

which contradicts supermodularity.

Step 2: Suppose $x_0 \le x_1 \le \cdots \le x_n$. If, for $i = 1, ..., n$, every $\mathrm{IC}_{i,i-1}$ (local downward incentive-compatibility constraint) holds as an equality, then all IC's are satisfied. This is shown in two steps. First, if every $\mathrm{IC}_{i,i-1}$ holds as an equality, then, for every $k < i$, $\mathrm{IC}_{k,i}$ is satisfied. To see this:

$$
\begin{aligned}
& t_i - t_k \\
= \;& (t_i - t_{i-1}) + (t_{i-1} - t_{i-2}) + \cdots + (t_{k+1} - t_k) \\
= \;& (u_1(x_i, \theta_i) - u_1(x_{i-1}, \theta_i)) \\
& + (u_1(x_{i-1}, \theta_{i-1}) - u_1(x_{i-2}, \theta_{i-1})) \\
& + \cdots + (u_1(x_{k+1}, \theta_{k+1}) - u_1(x_k, \theta_{k+1})) \\
\ge \;& (u_1(x_i, \theta_k) - u_1(x_{i-1}, \theta_k)) \\
& + (u_1(x_{i-1}, \theta_k) - u_1(x_{i-2}, \theta_k)) \\
& + \cdots + (u_1(x_{k+1}, \theta_k) - u_1(x_k, \theta_k)) \\
= \;& u_1(x_i, \theta_k) - u_1(x_k, \theta_k),
\end{aligned}
$$

where the second equality is due to the $\mathrm{IC}_{i,i-1}$ and the inequality is supermodularity. Second, if every $\mathrm{IC}_{i,i-1}$ holds as an equality, then, for every $k < i$, IC$_{i,k}$ is satisfied. This is because:

$$
\begin{aligned}
& t_i - t_k \\
= \;& (t_i - t_{i-1}) + (t_{i-1} - t_{i-2}) + \cdots + (t_{k+1} - t_k) \\
= \;& (u_1(x_i, \theta_i) - u_1(x_{i-1}, \theta_i)) \\
& + (u_1(x_{i-1}, \theta_{i-1}) - u_1(x_{i-2}, \theta_{i-1})) \\
& + \cdots + (u_1(x_{k+1}, \theta_{k+1}) - u_1(x_k, \theta_{k+1})) \\
\leq \;& (u_1(x_i, \theta_i) - u_1(x_{i-1}, \theta_i)) \\
& + (u_1(x_{i-1}, \theta_i) - u_1(x_{i-2}, \theta_i)) \\
& + \cdots + (u_1(x_{k+1}, \theta_i) - u_1(x_k, \theta_i)) \\
= \;& u_1(x_i, \theta_i) - u_1(x_k, \theta_i).
\end{aligned}
$$

Step 3: "If" part. Suppose $x_0 \leq x_1 \leq \cdots \leq x_n$ and construct a t such that every IC$_{i,i-1}$ holds as an equality. Let $t_0^* = 0$ and, for $i = 1, ...n$,

$$t_i^* = t_{i-1}^* + u_1(x_i, \theta_i) - u_1(x_{i-1}, \theta_i).$$

This t satisfies all local downward IC's by constraint as equalities, and, by Step 2, all IC's. ■

2. Profit Maximization

Having characterized the set of quantity vectors that can be implemented, we move to profit maximization.

Proposition 13 *Under A1, an optimal contract that implements x takes the form*

$$t_i^*(x) = t_{i-1}^*(x) + u_1(x_i, \theta_i) - u_1(x_{i-1}, \theta_i) \quad \textit{for } i = 1, ..., n.$$

Proof. Fix t and suppose the condition in the proposition is not satisfied. Start with the lowest i for which $t_i < t_{i-1} + u_1(x_i, \theta_i) - u_1(x_{i-1}, \theta_i)$ (the opposite inequality would violate $\mathsf{IC}_{i,i-1}$) and let $\tilde{t}_i = t_{i-1} + u_1(x_i, \theta_i) - u_1(x_{i-1}, \theta_i)$ and, for all $j > i$, let $\tilde{t}_j = t_j + \tilde{t}_i - t_i$. Repeat this procedure until all $\mathsf{IC}_{i,i-1}$ hold as equality. The new t is implementable by Proposition 12 and, as t is higher and x is unchanged, yields a strictly higher profit. ■

With Propositions 12 and 13, the principal's problem rewrites as

$$\max_x \sum_i p_i \left(u_0\left(x_i, \theta_i\right) + t_i^*(x)\right)$$
$$\text{subject to } 0 \leq x_1 \leq \cdots \leq x_n$$

The "relaxed" version of this problem is

$$\max_x \sum_i p_i \left(u_0\left(x_i, \theta_i\right) + t_i^*(x)\right)$$
$$\text{subject to } x \text{ nonnegative}$$

The advantage of the relaxed problem is that both the objective function and the constraint are separable in i. The problem can be split in n subproblems that can be solved separately.

In general, the solution to the relaxed problem can be different from the solution to the full problem. However, we can provide a condition under which the relaxed problem always has a solution that is nondecreasing in i. As that solution also solves the full problem, we can focus without loss of generality on the relaxed problem. Let $P_i = \sum_{j=i}^n p_j$ and define, for every x and i,

$$\phi(x, i) = u_0\left(x, \theta_i\right) + \frac{P_i}{p_i} u_1\left(x, \theta_i\right) - \frac{P_{i+1}}{p_i} u_1\left(x, \theta_{i+1}\right).$$

Proposition 14 *Under A1, if ϕ is supermodular in x and i, the relaxed problem has a solution that satisfies* $0 \leq x_1 \leq \cdots \leq x_n$.

Proof. First, notice that

$$
\begin{aligned}
& \sum_i p_i \left(u_0\left(x_i,\theta_i\right)+t_i^*(x)\right) \\
= & \left(p_1 u_0\left(x_1,\theta_1\right)+p_1 u_1\left(x_1,\theta_1\right)-p_1 u_1\left(x_0,\theta_1\right)\right) \\
& +\left(p_2 u_0\left(x_2,\theta_2\right)+p_2 u_1\left(x_1,\theta_1\right)-p_2 u_1\left(x_0,\theta_1\right)\right. \\
& \left.+p_2 u_1\left(x_2,\theta_2\right)-p_2 u_1\left(x_1,\theta_2\right)\right) \\
& +\cdots \\
= & -\sum_{j=1}^n p_j u_1\left(x_0,\theta_1\right) \\
& +p_1 u_0\left(x_1,\theta_1\right)+\sum_{j=1}^n p_j u_1\left(x_1,\theta_1\right)-\sum_{j=2}^n p_j u_1\left(x_1,\theta_2\right) \\
& +p_2 u_0\left(x_2,\theta_2\right)+\sum_{j=2}^n p_j u_1\left(x_2,\theta_2\right)-\sum_{j=2}^n p_j u_1\left(x_2,\theta_2\right) \\
& +\cdots \\
= & -\sum_{j=1}^n p_j u_1\left(x_0,\theta_1\right)+\sum_i p_i u_0\left(x_i,\theta_i\right) \\
& +\sum_{i=1}^n P_i u_1\left(x_i,\theta_i\right)-\sum_{i=1}^n P_{i+1} u_1\left(x_i,\theta_{i+1}\right) \\
= & -\sum_{j=1}^n p_j u_1\left(x_0,\theta_1\right)+\sum_i p_i \phi(x_i,i).
\end{aligned}
$$

Claim: Suppose that Y is a finite subset of the real line and that $f : Y \times \Re \rightarrow \Re$ is supermodular. Then, $Y^*(z) \equiv \arg\max_y f(y,z)$ is nondecreasing in z (given two finite real sets Y'' and Y' we say that $Y'' \geq Y'$ if $\max Y'' \geq \max Y'$ and $\min Y'' \geq \min Y'$).

Proof of the claim: Suppose that $z'' > z'$ but $\max Y^*(z'') < \max Y^*(z')$. As $Y^*(z'')$ and $Y^*(z')$ are maximizers,

$$f(\max Y^*(z''), z'') \geq f(\max Y^*(z'), z'')$$

and

$$f(\max Y^*(z'), z') \geq f(\max Y^*(z''), z'),$$

which combined with supermodularity implies

$$\begin{aligned} & f(\max Y^*(z''), z'') - f(\max Y^*(z''), z') \\ = \; & f(\max Y^*(z'), z'') - f(\max Y^*(z'), z'). \end{aligned}$$

As $f(\max Y^*(z'), z') \geq f(\max Y^*(z''), z')$, it must be that $f(\max Y^*(z'), z'') \geq f(\max Y^*(z''), z'')$, ie, $\max Y^*(z') \in Y^*(z'')$: a contradiction because we had assumed $\max Y^*(z'') < \max Y^*(z')$. The proof for $\min Y^*(z)$ is analogous.

The problem

$$\max_x \sum_i p_i \left(u_0\left(x_i, \theta_i\right) + t_i^*(x)\right)$$

rewrites as

$$\max_x \sum_i p_i \phi(x_i, i) = \sum_i p_i \max_{x_i} \phi(x_i, i).$$

By the claim, this problem has a nondecreasing solution: $x_1^* \leq \cdots \leq x_n^*$. Then the solution of the relaxed problem

$$\max_x \sum_i p_i \phi(x_i, i)$$
subject to x nonnegative

is also nondecreasing in i, which means that the monotonicity constraint of the full problem is not binding. A solution to the relaxed problem is a solution to the full problem. ■

Sufficient conditions for ϕ to be supermodular can be provided. Suppose that:

$$\frac{p_i}{P_{i+1}} \text{ is nondecreasing in } i \text{ for } i = 1, ..., n-1 \tag{A2}$$

$$u_0 \text{ is supermodular in } x \text{ and } \theta \tag{A3}$$

$$u_1(x, \theta_i) - u_1(x, \theta_{i+1}) \text{ is supermodular in } x \text{ and } i \tag{A4}$$

A2 requires that the hazard rate be nondecreasing in the type. A3 is the same of A1 but for the principal's utility. If u_1 is three-time differentiable, A4 is equivalent to $\frac{\partial^3 u_1}{\partial \theta^2 \partial x} \leq 0$ everywhere.

Proposition 15 *If A1 through A4 are satisfied, then ϕ is supermodular in x and i.*

Proof. The following two results are useful:

Claim 1: if $f(y, z)$ and $g(y, z)$ are supermodular functions, then $f(y, z) + g(y, z)$ is supermodular.

Claim 2: if $f(y, z)$ is supermodular in y and z and nonincreasing in y, and $g(z)$ is nonnegative and nonincreasing in z, then $f(y, z)g(z)$ is supermodular in y and z.

Proof of Claim 2: As f is nonincreasing in y and g is nonincreasing in z.

$$\left(f(y'', z'') - f(y', z'')\right)\left(g(z'') - g(z')\right) \geq 0$$

Hence,

$$\begin{aligned} & \left(f(y'', z'') - f(y', z'')\right) g(z'') \\ \geq & \left(f(y'', z'') - f(y', z'')\right) g(z') \\ \geq & \left(f(y'', z') - f(y', z')\right) g(z') \end{aligned}$$

where the seond inequality is due to the fact that f is supermodular and g is nonnegative.

Rewrite:

$$\begin{aligned} & \phi(x, i) \\ = & u_0(x, \theta_i) + \frac{P_{i+1}}{p_i}\left(u_1(x, \theta_i) - u_1(x, \theta_{i+1})\right) + u_1(x, \theta_i \end{aligned}$$

The first addend is supermodular in x and i (because θ_i is increasing in i). The second addend is the product of $\frac{P_{i+1}}{p_i}$, which is nonnegative and nonincreasing in i (because of A2), and $u_1(x, \theta_i) - u_1(x, \theta_{i+1})$, which is nonincreasing (by A1) and supermodular in x and i (by A4). The third addend is supermodular

by assumption. By Claim 1, $\phi(x, i)$ is supermodular.
■

The results presented so far are summarized by

Proposition 16 *Under A1 through A4, the principal's problem has a solution that satisfies:*

$\hat{x}_i \in \arg\max_x p_i u_0(x, \theta_i) + P_i u_1(x, \theta_i) - P_{i+1} u_1(x, \theta_{i+1})$

subject to $x \geq 0$ *for* $i = 1, ..., n$

$$\begin{aligned} \hat{t}_i &= \hat{t}_{i-1} + u_1(\hat{x}_i, \theta_i) - u_1(\hat{x}_{i-1}, \theta_i) \quad \textit{for } i = 1, ..., n \\ \hat{x}_0 &= 0 \\ \hat{t}_0 &= 0 \end{aligned}$$

In practice, one starts by computing $\left(\hat{x}, \hat{t}\right)$. If $\left(\hat{x}, \hat{t}\right)$ is unique, then it is a solution of the principal's problem. If there are multiple $\left(\hat{x}, \hat{t}\right)$, then there is at least one such that that $\hat{x}$ is nondecreaing in i, and that is a solution to the principal's problem.

Mechanism Design with Multiple Agents

- Implementability
  - Bayesian vs. dominant
  - Groves mechanisms
  - Myerson-Satterthwaite
- Correlated types

Implementability

Assumptions:

(B1) θ_i, θ_j independent. $\theta_i \sim P_i$. with strictly positive and differentiable density p_i.

(B2) Private values: $u_i(x, t_i, \theta_i)$ (and not $u_i(x, t_i, \theta)$), except possibly the principal.

(B3) Quasilinear preferences. Agent

$$u_i(x, t_i, \theta_i) = v_i(x, \theta_i) + t_i$$

Principal:

$$u_0(x, t_0, \theta_0) = v_0(x, \theta_0) - \sum_{i=1}^{I} t_i$$

or

$$u_0(x, t_0, \theta_0) = \sum_{i=0}^{I} v_i(x, \theta_i)$$

Allocation $x(\theta)$ is efficient if $x(\theta) \in \arg\max_x \sum_{i=0}^{I} v_i(x, \theta_i)$

Bayesian/Dominant Implementation

Bayesian: It is a Bayesian equilibrium for the agents to play according to the principal's wishes.

Dominant: It is a dominant strategy for each agent to play according to the principal's wishes, independent of what others do.

Two Revelation Principles:

(IC) For all $i, \theta_i, \hat{\theta}_i$,

$$E_{\theta_{-i}}\left[u_i\left(y\left(\theta_i, \theta_{-i}\right), \theta_i\right)\right] \geq E_{\theta_{-i}}\left[u_i\left(y\left(\hat{\theta}_i, \theta_{-i}\right), \theta_i\right)\right]$$

(DIC) For all $i, \theta_i, \hat{\theta}_i, \hat{\theta}_{-i}$,

$$u_i\left(y\left(\theta_i, \hat{\theta}_{-i}\right), \theta_i\right) \geq u_i\left(y\left(\hat{\theta}_i, \hat{\theta}_{-i}\right), \theta_i\right)$$

Obviously, (DIC) implies (IC). Mookherjee-Reichelstein identify conditions under which imposing (DIC) instead of (IC) involves no welfare loss.

Groves Mechanism

Any efficient x can be implemented in dominant strategies.

Idea: externality payments

Δ i's transfers $=$ Δ -i's payoff

Suppose $x^*(\theta) \in \arg\max_x \sum_{i=0}^{I} v_i(x, \theta_i)$

We want to implement x^*. Let

$$t_i^*\left(\hat{\theta}\right) = \sum_{j \neq i} v_j\left(x^*\left(\hat{\theta}_i, \hat{\theta}_{-i}\right), \hat{\theta}_j\right) + \tau_i\left(\hat{\theta}_{-i}\right),$$

where τ_i is an arbitrary function.

Suppose (DIC) is not satisfied, ie there exists $\theta, \hat{\theta}_{-i}, \hat{\theta}_i \neq \theta_i$ such that

$$\begin{aligned} & v_i\left(x^*\left(\hat{\theta}_i, \hat{\theta}_{-i}\right), \theta_i\right) + t_i^*\left(\hat{\theta}_i, \hat{\theta}_{-i}\right) \\ > \; & v_i\left(x^*\left(\theta_i, \hat{\theta}_{-i}\right), \theta_i\right) + t_i^*\left(\theta_i, \hat{\theta}_{-i}\right) \end{aligned}$$

Then,

$$\begin{aligned}
& v_i\left(x^*\left(\hat{\theta}_i, \hat{\theta}_{-i}\right), \theta_i\right) + \sum_{j \neq i} v_j\left(x^*\left(\hat{\theta}_i, \hat{\theta}_{-i}\right), \hat{\theta}_j\right) \\
> \ & v_i\left(x^*\left(\theta_i, \hat{\theta}_{-i}\right), \theta_i\right) + \sum_{j \neq i} v_j\left(x^*\left(\theta_i, \hat{\theta}_{-i}\right), \hat{\theta}_j\right)
\end{aligned}$$

Let $\tilde{\theta} = \left(\theta_i, \hat{\theta}_{-i}\right)$ and $x^*\left(\tilde{\theta}\right) \in \arg\max_x \sum_{i=0}^{I} v_i\left(x, \tilde{\theta}_i\right)$. Then, for all x,

$$\begin{aligned}
& v_i\left(x^*\left(\theta_i, \hat{\theta}_{-i}\right), \theta_i\right) + \sum_{j \neq i} v_j\left(x^*\left(\theta_i, \hat{\theta}_{-i}\right), \hat{\theta}_j\right) \\
\geq \ & v_i\left(x, \theta_i\right) + \sum_{j \neq i} v_j\left(x, \hat{\theta}_j\right)
\end{aligned}$$

So, if $x = x^*\left(\hat{\theta}_i, \hat{\theta}_{-i}\right)$,

$$\begin{aligned}
& v_i\left(x^*\left(\theta_i, \hat{\theta}_{-i}\right), \theta_i\right) + \sum_{j \neq i} v_j\left(x^*\left(\theta_i, \hat{\theta}_{-i}\right), \hat{\theta}_j\right) \\
\geq \ & v_i\left(x^*\left(\hat{\theta}_i, \hat{\theta}_{-i}\right), \theta_i\right) + \sum_{j \neq i} v_j\left(x^*\left(\hat{\theta}_i, \hat{\theta}_{-i}\right), \hat{\theta}_j\right)
\end{aligned}$$

Contradiction.

Example 2: Second-Price Auction

A principal must allocate a good among n agents.

Agent i's valuation for the good is θ_i. Valuations are independently distributed among buyers.

Allocation: $\left(i^*, \{t_i\}_{i \in I}\right)$

Agents' payoff

$$v_i\left(x, t_i, \theta_i\right) = \begin{cases} \theta_i + t_i & \text{if} \quad i = i^* \\ t_i & \text{if} \quad i \neq i^* \end{cases}$$

Principal's payoff:

$$v_0\left(x, t_i, \theta_i\right) = -\sum_{i=1}^{I} t_i$$

Find Groves mechanism:

1. Efficient allocation

$$\sum_{i=0}^{I} v_i\left(x, t_i, \theta_i\right) = \theta_{i^*}$$

Maximize surplus by selecting

$$i^* = \arg\max_i \theta_i$$

2. Externality transfers

$$\sum_{j \neq i} v_j\left(x^*\left(\hat{\theta}_i, \hat{\theta}_{-i}\right), \hat{\theta}_j\right) = \begin{cases} 0 & \text{if } \; i = i^* \\ \max_{j \neq i} \hat{\theta}_j & \text{if } \; i \neq i^* \end{cases}$$

The transfers take the form

$$t_i(\hat{\theta}_i, \hat{\theta}_{-i}) = \begin{cases} 0 + \tau_i\left(\hat{\theta}_{-i}\right) & \text{if } \; \hat{\theta}_i = \max_j \hat{\theta}_j \\ \max_{j \neq i} \hat{\theta}_j + \tau_i\left(\hat{\theta}_{-i}\right) & \text{if } \; \hat{\theta}_i \neq \max_j \hat{\theta}_j \end{cases}$$

3. If we fix $\tau_i\left(\hat{\theta}_{-i}\right) = -\max_{j \neq i} \hat{\theta}_j$, we obtain

$$t_i(\hat{\theta}_i, \hat{\theta}_{-i}) = \begin{cases} -\max_{j \neq i} \hat{\theta}_j & \text{if } \; \hat{\theta}_i > \max_{j \neq i} \hat{\theta}_j \\ 0 & \text{if } \; \hat{\theta}_i < \max_{j \neq i} \hat{\theta}_j \end{cases}.$$

which is the standard second-price auction.

Inefficiency with Budget Balance

Myerson-Satterthwaite 1983

Coase Theorem: with complete contracts, two parties must achieve an efficient allocation.

With asymmetric info this is not true.

Example: Two-agent trading game: $x \in \{0, 1\}$

seller	$\theta_1^L = -5$	$\theta_1^H = 0$
buyer	$\theta_2^L = 1$	$\theta_2^H = 6$

with $p_1^H = p_2^H = \frac{1}{2}$.

Suppose a benevolent mechanism designer maximizes surplus (and hence efficiency). Then $x^*(\theta_1, \theta_2)$ is

	θ_2^L	θ_2^H
θ_1^L	$x = 0$	$x = 1$
θ_1^H	$x = 1$	$x = 1$

Without budget balance, we can use a Groves mechanism:

$$\begin{aligned} t_1(\theta_1, \theta_2) &= x^*(\theta_1, \theta_2)\,\theta_2 + \tau_1(\theta_2) \\ t_2(\theta_1, \theta_2) &= x^*(\theta_1, \theta_2)\,\theta_1 + \tau_2(\theta_1) \end{aligned}$$

where t_1 (t_2) is the monetary transfer *to* the seller (buyer). So,

$$t_1(\theta_1, \theta_2) = \begin{cases} 0 + \tau_1^L & \text{if } \theta_1 = -5, \theta_2 = 1 \\ 6 + \tau_1^H & \text{if } \theta_1 = -5, \theta_2 = 6 \\ 1 + \tau_1^L & \text{if } \theta_1 = 0, \theta_2 = 1 \\ 6 + \tau_1^H & \text{if } \theta_1 = 0, \theta_2 = 6 \end{cases}$$

$$t_2(\theta_1, \theta_2) = \begin{cases} 0 + \tau_2^L & \text{if } \theta_1 = -5, \theta_2 = 1 \\ -5 + \tau_2^L & \text{if } \theta_1 = -5, \theta_2 = 6 \\ 0 + \tau_2^H & \text{if } \theta_1 = 0, \theta_2 = 1 \\ 0 + \tau_2^H & \text{if } \theta_1 = 0, \theta_2 = 6 \end{cases}$$

Total payments are

$$t_1 + t_2 = \begin{cases} 0 + \tau_1^L + \tau_2^L & \text{if } \theta_1 = -5, \theta_2 = 1 \\ 1 + \tau_1^H + \tau_2^L & \text{if } \theta_1 = -5, \theta_2 = 6 \\ 1 + \tau_1^L + \tau_2^H & \text{if } \theta_1 = 0, \theta_2 = 1 \\ 6 + \tau_1^H + \tau_2^H & \text{if } \theta_1 = 0, \theta_2 = 6 \end{cases}$$

$\Longrightarrow$ there does not exist a Groves mechanism that satisfies budget balance.

Is there *any* mechanism that implements x^* and satisfies BB?

$$x_1^H = E_{\theta_2}\left(x^*\left(\theta_1^H, \theta_2\right)\right) = 1 \quad x_2^H = 1$$
$$x_1^L = E_{\theta_2}\left(x^*\left(\theta_1^L, \theta_2\right)\right) = \tfrac{1}{2} \quad x_2^L = \tfrac{1}{2}$$

Show that IR_1^L, IR_2^L, IC_1^H, IC_2^H, and BB are inconsistent.

IR_1^L $\theta_1^L x_1^L + t_1^L \geq 0$

$$\Longrightarrow -5\tfrac{1}{2} + t_1^L \geq 0$$

$$\Longrightarrow t_1^L \geq 2.5$$

(min price for seller with high-quality good)

IR_2^L $\theta_2^L x_2^L + t_2^L \geq 0$

$$\Longrightarrow 1\tfrac{1}{2} + t_2^L \geq 0$$

$$\Longrightarrow t_2^L \geq -0.5$$

(max price for buyer with low demand)

$\text{IC}_1^H \quad \theta_1^H x_1^H + t_1^H \geq \theta_1^H x_1^L + t_1^L$

$$\Longrightarrow 0 \cdot 1 + t_1^H \geq 0\tfrac{1}{2} + t_1^L$$

$$\Longrightarrow t_1^H \geq t_1^L$$

(min price for seller with low-quality good)

$\text{IC}_2^H \quad \theta_2^H x_2^H + t_2^H \geq \theta_2^H x_2^L + t_2^L$

$$\Longrightarrow 6 \cdot 1 + t_2^H \geq 6\tfrac{1}{2} + t_2^L$$

$$\Longrightarrow t_2^H \geq t_2^L - 3$$

(max price for buyer with high demand)

By BB, let $t(\theta_1, \theta_2) = t_1(\theta_1, \theta_2) = -t_2(\theta_1, \theta_2)$. Then,

$$\begin{aligned}
t_1^L &= \frac{1}{2}(t(-5,1) + t(-5,6)) \\
t_1^H &= \frac{1}{2}(t(0,1) + t(0,6)) \\
t_2^L &= \frac{1}{2}(t(-5,1) + t(0,1)) \\
t_2^H &= \frac{1}{2}(t(-5,6) + t(0,6))
\end{aligned}$$

Rewrite

IR_1^L $t(-5,1)+t(-5,6)\geq 5$

IR_2^L $-t(-5,1)-t(0,1)\geq -1$

$\mathsf{IR}_1^L+\mathsf{IR}_2^L$ $t(-5,6)-t(0,1)\geq 4$

and

IC_1^H $t(0,1)+t(0,6)\geq t(-5,1)+t(-5,6)$

IC_2^H $-t(-5,6)-t(0,6)\geq -t(-5,1)-t(0,1)-6$

$\mathsf{IC}_1^H+\mathsf{IC}_2^H$ $2\,(t(0,1)-t(-5,6))\geq -6$

Put together

$$\begin{aligned} t(-5,6)-t(0,1) &\geq 4 \\ t(-5,6)-t(0,1) &\leq 3 \end{aligned}$$

$\Longrightarrow$ contradiction.

The example generalizes

Theorem 17 (Myerson-Satterthwaite) *Suppose that θ_1 and θ_2 have differentiable, strictly positive densities on $\left[\theta_1^L, \theta_1^H\right]$ and $\left[\theta_2^L, \theta_2^H\right]$, and that both $x^* = 1$ and $x^* = 0$ have positive probability. Then, there is no mechanism that satisfies IC, IR, and BB.*

Correlated Types

Cremer-McLean (1985), risk neutrality.

So far we have assumed that θ's are independent across agents.

Suppose instead they are perfectly correlated: θ_1, θ_2. Then there is a "shoot-the-liar" mechanism:

$$t\left(\hat{\theta}_1, \hat{\theta}_2\right) = -\infty \text{ if } \hat{\theta}_1 \neq \hat{\theta}_2$$

There exists a Bayesian equilibrium that induces full revelation at no cost for the principal.

Imperfect correlation?

Take $p\left(\theta_i | \theta_{-i}\right)$

Full rank: Suppose $\left\{p\left(\theta_i | \theta_{-i}\right)\right\}_{\theta_{-i} \in \Theta_{-i}}$ are linearly independent

$\Longrightarrow \theta_i$ is informative on θ_{-i}.

Cremer-McLean: Under risk neutrality and full rank, the principal can implement any x without leaving any rent to the agent.

Example:

Reconsider the auction with two types.

Suppose $\theta_i \in \{L, H\}$ with equal probability, and

$$\begin{aligned} \Pr\left(\theta_i = H | \theta_j = H\right) &= \rho \\ \Pr\left(\theta_i = H | \theta_j = L\right) &= 1 - \rho \end{aligned}$$

with $\rho \in \left(\frac{1}{2}, 1\right]$.

The principal offers the following mechanism;

- First there is a bet: Agent $i = 1, 2$ reports θ_i. He receives a payment of M if $\theta_i = \theta_j$ and he pays $\frac{\rho}{1-\rho} M$ if $\theta_i \neq \theta_j$.

- Next, the principal sells the good to the agent with the higher θ_i at price θ_i.

If both agents tell the truth, the bet has an expected value of zero:

$$\begin{aligned} & \mathsf{Pr}\left(\theta_i = \theta_j\right) M - \mathsf{Pr}\left(\theta_i \neq \theta_j\right) \frac{\rho}{1-\rho} M \\ = \ & \rho M - \rho M = 0 \end{aligned}$$

If one agent lies, the bet has a negative expected value

$$\begin{aligned} & \mathsf{Pr}\left(\theta_i \neq \theta_j\right) M - \mathsf{Pr}\left(\theta_i = \theta_j\right) \frac{\rho}{1-\rho} M \\ = \ & (1-\rho) M - \rho \frac{\rho}{1-\rho} M = -\frac{2\rho - 1}{1-\rho} M \end{aligned}$$

By choosing an appropriately large M, the principal makes sure that the agent never lies.

Note that the PC is satisfied.

This does not work if $\rho = \frac{1}{2}$.

Drawbacks of this approach?

Lecture 4: Dynamic Games of Incomplete Information

Combine subgame perfection and Bayesian equilibrium.

Main challenge: The action a player chooses can signal her type to players who move afterwards. The player realizes the signalling component and factors it into her choice. The following players realize that she realizes and...

Road Map

- The basic signalling game.

- Definition of perfect Bayesian equilibrium

- Examples:

  1. Reputation game
     Distinction between separating, semi-separating, and pooling
     Method for checking existence of PBE

  2. Spence's job signalling
     Multiple equilibria
     Intuitive Criterion

  3. Money burning
     Single crossing condition
     Advertising

- Generalization

The Basic Signalling Game

Player 1: sender, has private info $\theta \in \Theta$.

Player 2: receiver, no private info.

Prior distribution p on Θ.

1. The sender observes θ and selects $a_1 \in A_1$;

2. The receiver observes a_1 and selects $a_2 \in A_2$.

Payoffs: $u_1(a_1, a_2, \theta)$ and $u_2(a_1, a_2, \theta)$

Perfect Bayesian Equilibrium

- Mixed strategy for player 1 (distribution over A_1 given θ): $\sigma_1(\cdot|\theta)$.

- Belief for player 2 (distribution over Θ given a_1): $\mu(\cdot|a_1)$.

- Mixed strategy for player 2 (distribution over A_2 given a_1): $\sigma_2(\cdot|a_1)$.

A perfect Bayesian equilibrium (PBE) is a strategy profile $\left(\sigma_1^*, \sigma_2^*\right)$ and a belief μ such that

- *Player 1's strategy is optimal*. For any type, the action that the sender plays maximizes her expected payoff given the receiver's equilibrium play:

$$\forall\theta, \forall a_1^* \text{ such that } \sigma_1^*\left(a_1^*|\theta\right) > 0,$$

$$a_1^* \in \arg\max_{a_1} \sum_{a_2} \sigma_2^*\left(a_2|a_1\right) u_1\left(a_1, a_2, \theta\right).$$

- *Player 2's belief is consistent*. For every action that the sender plays with positive probability, the receiver derives his belief on the sender's type using Bayes' theorem:

$$\forall a_1^* \text{ such that } \sum_{\theta'\in\Theta} p\left(\theta'\right)\sigma_1^*\left(a_1^*|\theta'\right) > 0,$$

$$\mu\left(\theta|a_1^*\right) = \frac{p\left(\theta\right)\sigma_1^*\left(a_1^*|\theta\right)}{\sum_{\theta'\in\Theta} p\left(\theta'\right)\sigma_1^*\left(a_1^*|\theta'\right)}.$$

- *Player 2's strategy is optimal*. For every action the sender could play, the receiver's action is optimal given his belief:

$$\forall a_1, \forall a_2^* (a_1) \text{ such that } \sigma_2^* (a_2^* (a_1) | a_1) > 0,$$

$$a_2^* (a_1) \in \arg\max_{a_2} \sum_{\theta} \mu (\theta | a_1) u_2 (a_1, a_2, \theta) .$$

Out-of-equilibrium beliefs:

The consistency requirement applies only to actions that are played with positive probability in equilibrium.

Suppose in a certain equilibrium action a_1 is played with probability zero. Then, no restriction is imposed on belief $\mu (\theta | a_1)$.

Finding PBE's is a form of art. Practice, practice, practice.

1. Reputation Game

Kreps-Wilson (1982), Milgrom-Roberts (1982).

Two firms: incumbent ($i = 1$), challenger ($i = 2$).
Two periods.

1. Incumbent observes $\theta \in \{\text{sane,crazy}\}$, where $\Pr(\text{sane}) = p$. She chooses $a_1 \in \{\text{prey,accomodate}\}$.

2. Challenger chooses $a_2 \in \{\text{stay,exit}\}$.

Payoffs are

sane

	stay	exit
prey	P_1, P_2	$M_1, 0$
accomodate	D_1, D_2	$M_1, 0$

crazy

	stay	exit
prey	K_1, K_2	$K_1, 0$
accomodate	D_1, D_2	$M_1, 0$

With

$$
\begin{aligned}
P_1 &< D_1 < M_1 < K_1; \\
K_2 &< 0 < P_2 < D_2.
\end{aligned}
$$

Also assume that the proportion of crazy types is not too high:

$$pP_2 \geq -(1-p)\,K_2 \tag{5}$$

What is the set of PBE of this game?

The crazy type has a dominant strategy. He should always prey: $a_1^*(\text{crazy}) = \text{prey}$.

What does the sane type do? Let

$$\alpha = \Pr\left(a_1^*(\text{sane}) = \text{prey}\right).$$

Reduced form of a game of entry deterrence.

- The incumbent monopolist faces a challenger.
- Some crazy monopolists love to fight on price, but the sane majority would rather accomodate entry than engage in a long-term fight.
- The challenger enters the market without knowing what type the monopolist is.
- The incumbent can engage in a price battle (prey) or collude (accomodate). Some sane monopolists may want to pretend to be crazy.
- The challenger can stay or leave. If the challenger stays, a crazy incumbent keeps fighting while a sane incumbent colludes.

More in general: this is a theory of threats.

Three Classes of Equilibria

1. Separating equilibrium: $\alpha = 0$. A sane challenger and a crazy challenger always take different actions. In equilibrium the challenger knows the incumbent's type.

2. Semi-separating equilibrium: $\alpha \in (0, 1)$. A sane type and a crazy type sometimes choose different actions, but not always.

3. Pooling equilibrium: $\alpha = 1$. The two types always choose the same action.

Separating Equilibrium

Proposition 18 *There exists no separating equilibrium.*

Strategy of proof:

1. Assume there exists a separating equilibrium (the sane incumbent accomodates: $\alpha = 0$).

2. Determine the challenger's belief, determine the challenger's optimal strategy.

3. Determine the incumbent's optimal strategy.

4. Show that the incumbent's optimal strategy is not to accomodate.

If $\alpha = 0$, the *challenger's belief* is

$$\forall a_1^* \text{ such that } \sum_{\theta' \in \Theta} p\left(\theta'\right) \sigma_1^* \left(a_1^* | \theta'\right) > 0,$$

$$\mu\left(\theta | a_1^*\right) = \frac{p\left(\theta\right) \sigma_1^* \left(a_1^* | \theta\right)}{\sum_{\theta' \in \Theta} p\left(\theta'\right) \sigma_1^* \left(a_1^* | \theta'\right)}.$$

That is,

$$\begin{aligned} \mu\left(\theta = \text{sane} | \text{prey}\right) &= 0; \\ \mu\left(\theta = \text{sane} | \text{accomodate}\right) &= 1. \end{aligned}$$

There are no out-of-equilibrium beliefs.

The *challenger's optimal strategy* is:

$$\forall a_1, \forall a_2^*(a_1) \text{ such that } \sigma_2^*(a_2^*|a_1) > 0,$$

$$a_2^*(a_1) \in \arg\max_{a_2} \sum_\theta \mu(\theta|a_1)\, u_2(a_1, a_2, \theta).$$

Suppose $a_1 = \text{prey}$. The expected payoff is

$$\begin{aligned} & \sum_\theta \mu(\theta|\text{prey})\, u_2(\text{prey}, a_2, \theta) \\ = \; & u_2(\text{prey}, a_2, \text{crazy}) \end{aligned}$$

and

$$u_2(\text{prey}, a_2, \text{crazy}) = \begin{cases} K_2 & \text{if } a_2 = \text{stay}; \\ 0 & \text{if } a_2 = \text{exit}. \end{cases}$$

The optimal strategy is to exit.

Suppose $a_1 =$ accomodate. The expected payoff is

$$\begin{aligned} & \sum_\theta \mu\left(\theta | \text{accomodate}\right) u_2\left(\text{accomodate}, a_2, \theta\right) \\ = \;& u_2\left(\text{accomodate}, a_2, \text{sane}\right) \end{aligned}$$

and

$$u_2\left(\text{accomodate}, a_2, \text{sane}\right) = \begin{cases} D_2 & \text{if } a_2 = \text{stay;} \\ 0 & \text{if } a_2 = \text{exit.} \end{cases}$$

The optimal strategy is to stay.

To recap, the challenger's optimal strategy is

$$a_2^*\left(a_1\right) = \begin{cases} \text{stay} & \text{if } a_1 = \text{accomodate;} \\ \text{exit} & \text{if } a_1 = \text{prey.} \end{cases}$$

The *incumbent's optimal strategy* is

$$\forall \theta, \forall a_1^* \text{ such that } \sigma_1^* \left(a_1^* | \theta\right) > 0,$$

$$a_1^* \in \arg\max_{a_1} \sum_{a_2} \sigma_2^* \left(a_2 | a_1\right) u_1 \left(a_1, a_2, \theta\right).$$

Given the challenger's strategy (which we saw above), the incumbent's expected payoff is

$$\begin{aligned} & \sum_{a_2} \sigma_2^* \left(a_2 | a_1\right) u_1 \left(a_1, a_2, \theta\right) \\ = & \begin{cases} u_1 \left(\text{accomodate}, \text{stay}, \theta\right) & \text{if } a_1 = \text{accomodate;} \\ u_1 \left(\text{prey}, \text{exit}, \theta\right) & \text{if } a_1 = \text{prey.} \end{cases} \end{aligned}$$

We already know that preying is optimal when $\theta =$ crazy.

Suppose $\theta =$ sane. The incumbent gets

$$\begin{array}{ll} u_1 \left(\text{accomodate}, \text{stay}, \text{sane}\right) = D_1 & \text{if } a_1 = \text{accomodate;} \\ u_1 \left(\text{prey}, \text{exit}, \text{sane}\right) = M_1 & \text{if } a_1 = \text{prey.} \end{array}$$

As $M_1 > D_1$, a sane incumbent's optimal strategy is to prey.

We have a *contradiction* because we assumed that a sane incumbent accomodates. There exists no separating equilibrium.

Semi-separating Equilibrium

Proposition 19 *There exists a unique semi-separating equilibrium. A sane incumbent preys with probability* $\alpha = -\frac{(1-p)K_2}{pP_2}$*. If the incumbent accomodates, the challenger stays. If the incumbent preys, the challenger stays with probability* $\frac{M_1-D_1}{M_1-P_1}$*.*

Strategy of proof: same as for the separating equilibrium but instead of a contradiction we get an equilibrium.

Suppose $\alpha \in (0, 1)$.

The *challenger's belief* is

$$\mu\,(\text{sane}|\text{prey}) = \frac{p\alpha}{p\alpha + 1 - p};$$
$$\mu\,(\text{sane}|\text{accomodate}) = 1.$$

There are no out-of-equilibrium beliefs.

The *challenger's optimal strategy* is:

The expected payoff is

$$\begin{aligned} & \sum_{\theta} \mu\left(\theta | a_1\right) u_2\left(a_1, a_2, \theta\right) \\ = \; & \mu\left(\text{sane} | a_1\right) u_2\left(a_1, a_2, \text{sane}\right) \\ & + \left(1 - \mu\left(\text{sane} | a_1\right)\right) u_2\left(a_1, a_2, \text{crazy}\right) \end{aligned}$$

Suppose $a_1 = \text{accomodate}$. If $a_2 = \text{stay}$, the expected payoff is D_2. If $a_2 = \text{exit}$, the payoff is 0. Therefore, the challenger should stay.

Suppose $a_1 = \text{prey}$. If $a_2 = \text{stay}$, the expected payoff is

$$\begin{aligned} & \mu\left(\text{sane} | \text{prey}\right) P_2 + \left(1 - \mu\left(\text{sane} | \text{prey}\right)\right) K_2 \\ = \; & \frac{p\alpha}{p\alpha + 1 - p} P_2 + \frac{1 - p}{p\alpha + 1 - p} K_2 \end{aligned}$$

If $a_2 = \text{exit}$, the expected payoff is 0. Therefore, the challenger stays if and only if

$$p\alpha P_2 \geq -\left(1 - p\right) K_2.$$

That is, the challenger

$$\begin{array}{ll} \text{exits} & \text{if } \alpha < -\frac{(1-p)K_2}{pP_2} \\ \text{is indifferent} & \text{if } \alpha = -\frac{(1-p)K_2}{pP_2} \\ \text{stays} & \text{if } \alpha > -\frac{(1-p)K_2}{pP_2} \end{array}$$

Note that $-\frac{(1-p)K_2}{pP_2} \in (0, 1)$ because of (5).

The *incumbent's optimal strategy*:

In order for $\alpha \in (0, 1)$, the incumbent's expected payoff must be the same when he preys and when he accomodates:

If the incumbent accomodates, the challenger stays and the payoff is D_1.

If the incumbent preys, let $\beta = \Pr(a_2(\text{prey}) = \text{ stay})$. The incumbent's expected payoff is

$$\beta P_1 + (1 - \beta) M_1.$$

The payoffs are equal if

$$\beta = \frac{M_1 - D_1}{M_1 - P_1} \in (0, 1).$$

But if $\beta \in (0, 1)$, it must be that

$$\alpha = -\frac{(1 - p) K_2}{p P_2}.$$

To re-cap: in a semi-separating equilibrium, it must be that

$$\begin{aligned}
\alpha &= -\frac{(1-p)\,K_2}{pP_2}; \\
\beta &= \frac{M_1 - D_1}{M_1 - P_1}.
\end{aligned}$$

For instance, suppose that $p = \frac{3}{4}$, $K_2 = -1$, $P_2 = 1$, $M_1 = 4$, $D_1 = 2$, and $P_1 = 1$. Then,

$$\alpha = \frac{1}{3} \quad \beta = \frac{2}{3} \quad \frac{p\alpha}{p\alpha + 1 - p} = \frac{1}{2}.$$

A sane incumbent preys with probability $\frac{1}{3}$.

If the incumbent preys, the challenger believes that she is sane with probability $\frac{1}{2}$.

If the incumbent preys, the challenger stays with probability $\frac{2}{3}$.

Pooling Equilibrium

Proposition 20 *There exists no pooling equilibrium.*

Strategy of proof:

1. Assume there exists a pooling equilibrium (the sane incumbent always preys: $\alpha = 1$).

2. Determine the challenger's belief (watch out for out-of-equilibrium beliefs).

3. Determine the challenger's optimal strategy for every possible out-of-equilibrium belief.

4. Show that the incumbent's optimal strategy is to accomodate.

Challenger's belief:

If $\alpha = 1$, the only action that the incumbent plays in equilibrium is $a_1 = \text{prey}$.

Belief consistency requires:

$$\mu(\theta|a_1^*) = \frac{p(\theta)\,\sigma_1^*(\text{prey}|\theta)}{\sum_{\theta' \in \Theta} p(\theta')\,\sigma_1^*(\text{prey}|\theta')} = \frac{p(\theta)}{\sum_{\theta' \in \Theta} p(\theta')} = p(\theta).$$

As both types play the same action, the belief equals the prior.

Instead, the belief $\mu(\theta|\text{accomodate})$ is out of equilibrium because $a_1^* = \text{accomodate}$ is never played.

Belief consistency imposes no requirement on $\mu(\theta|\text{accomoda}$

Let

$$\gamma = \mu(\theta|\text{accomodate}) \in [0, 1].$$

As we want to show that there exists no pooling equilibrium, we have to examine all possible values of γ.

Challenger's strategy:

If $a_1 =$ prey, the challenger gets

$$\begin{array}{ll} pP_2 + (1-p)\,K_2 & \text{if } a_2 = \text{stay;} \\ 0 & \text{if } a_2 = \text{exit.} \end{array}$$

The challenger stays because by (5), $pP_2+(1-p)\,K_2 > 0$.

If $a_1 =$ accomodate, the challenger's payoff is D_2 independent of whether $\theta =$ sane or $\theta =$ crazy. The challenger stays.

The payoff does not depend on the out-of-equilibrium belief γ. Lucky case.

The challenger's optimal strategy is to always stay.

Incumbent's strategy:

The incumbent knows that the challenger stays.

If $\theta =$ sane, the payoff is P_1 if $a_1 =$ prey and D_1 if $a_1 =$ accomodate. As $D_1 > P_1$, a sane incumbent acoomodates.

This is a contradiction because we had assumed that a sane incumbent always preys.

To Re-cap

Proposition 21 *The reputation game has a unique perfect Bayesian equilibrium. It is a semi-separating equilibrium in which a sane incumbent preys with probability* $\alpha = -\frac{(1-p)K_2}{pP_2}$.

2. Spence's Job Signalling Game

Spence (1974)

Players: a worker, a large number of firms

1. The worker observes her type $\theta \in \{\theta_L, \theta_H\}$ where $\theta_L < \theta_H$ and $\Pr(\theta = \theta_H) = \gamma$, and she chooses a level of education $a \in [0, \infty)$ at cost $\frac{a}{\theta}$.

2. The firms observe a and offer the worker a wage w.

3. The worker selects one of the firms and generates a product of value θ.

The payoff of the firm who hires the worker is

$$\theta - w.$$

The payoff of the worker is

$$w - \frac{a}{\theta}.$$

Idea: education is a sorting device (extreme case)

Reduced form of the game (assume firms have symmetric beliefs):

1. The worker observes her type $\theta \in \{\theta_L, \theta_H\}$ where $\theta_L < \theta_H$, and she chooses a level of education $a \in [0, \infty)$ at cost $\frac{a}{\theta}$.

2. The worker receives a wage

$$w = \mu(\theta_H|a)\,\theta_H + \mu(\theta_L|a)\,\theta_L,$$

where $\mu(\theta|a)$ is the firm's belief on the worker's type.

Denote the firm's belief as

$$\hat{\gamma}(a) = \mu(\theta_H|a)\,.$$

The wage is then

$$w = \theta_L + \hat{\gamma}(a)(\theta_H - \theta_L)\,.$$

Separating Equilibria

Suppose $a(\theta_L) \neq a(\theta_H)$.

Belief/wage on the equilibrium path:

$$\hat{\gamma}(a(\theta_L)) = 0 \quad \hat{\gamma}(a(\theta_H)) = 1.$$

Belief/wage off the equilibrium path:

$$\hat{\gamma}(a) \in [0,1] \quad \forall a \notin \{a(\theta_L), a(\theta_H)\}.$$

1. The bad worker should not want to deviate:

   $$\theta_L - \frac{a(\theta_L)}{\theta_L} \geq \theta_L + \hat{\gamma}(a)(\theta_H - \theta_L) - \frac{a}{\theta_L} \quad \forall a$$

   This implies that $a(\theta_L) = 0$. Suppose not. If not, the worker could get a higher payoff by playing $a = 0$.
   With $a(\theta_L) = 0$, the condition above re-writes as

   $$\hat{\gamma}(a) \leq \frac{a}{\theta_L(\theta_H - \theta_L)} \quad \forall a$$

2. The bad worker should not want to deviate to $a(\theta_H)$:

$$\theta_L \geq \theta_H - \frac{a(\theta_H)}{\theta_L} \qquad (6)$$

3. The good worker should not want to deviate to $a(\theta_L) = 0$:

$$\theta_H - \frac{a(\theta_H)}{\theta_H} \geq \theta_L \qquad (7)$$

   Conditions (6) and (7) together imply

$$\theta_L(\theta_H - \theta_L) \leq a(\theta_H) \leq \theta_H(\theta_H - \theta_L)$$

4. The good worker should not want to deviate to any other a:

$$\theta_H - \frac{a(\theta_H)}{\theta_H} \geq \theta_L + \hat{\gamma}(a)(\theta_H - \theta_L) - \frac{a}{\theta_H} \quad \forall a$$

   The condition rewrites as

$$\hat{\gamma}(a) \leq 1 - \frac{a(\theta_H) - a}{\theta_H(\theta_H - \theta_L)} \quad \forall a$$

Together with the condition on the bad worker this implies

$$\hat{\gamma}(a) \leq \min\left(\frac{a}{\theta_L(\theta_H - \theta_L)}, 1 - \frac{a(\theta_H) - a}{\theta_H(\theta_H - \theta_L)}\right)$$

To re-cap, the necessary and sufficient conditions for a separating equilibrium are

$$\begin{aligned} a(\theta_L) &= 0; \\ \theta_L(\theta_H - \theta_L) &\leq a(\theta_H) \leq \theta_H(\theta_H - \theta_L); \\ \hat{\gamma}(a) &\leq \min\left(\frac{a}{\theta_L(\theta_H - \theta_L)}, 1 - \frac{a(\theta_H) - a}{\theta_H(\theta_H - \theta_L)}\right) \end{aligned}$$

Note that the belief condition is satisfied by setting discontinuous beliefs of this form:

$$\hat{\gamma}(a) = \begin{cases} 0 & \text{if } a < a(\theta_H) \\ 1 & \text{if } a \geq a(\theta_H). \end{cases}$$

Example: Let $\theta_L = 1$ and $\theta_H = 2$.

The condition on $a(\theta_H)$ becomes $1 \leq a(\theta_H) \leq 2$.

For every $\bar{a} \in [1, 2]$, there exists a separating equilibrium in which

$$\begin{aligned} a(\theta_L) &= 0; \\ a(\theta_H) &= \bar{a}; \\ \hat{\gamma}(a) &= \begin{cases} 0 & \text{if } a < \bar{a} \\ 1 & \text{if } a \geq \bar{a}. \end{cases} \end{aligned}$$

Pooling Equilibria

Suppose $a(\theta_L) = a(\theta_H) = \bar{a}$.

Belief/wage on the equilibrium path:

$$\hat{\gamma}(\bar{a}) = \gamma.$$

Belief/wage off the equilibrium path:

$$\hat{\gamma}(a) \in [0, 1] \quad \forall a \neq \bar{a}.$$

1. The bad worker should not want to deviate:

   $$\gamma(\theta_H - \theta_L) - \frac{\bar{a}}{\theta_L} \geq \hat{\gamma}(a)(\theta_H - \theta_L) - \frac{a}{\theta_L} \quad \forall a$$

2. In particular, the bad worker should not want to deviate to $a = 0$:

   $$\gamma(\theta_H - \theta_L) - \frac{\bar{a}}{\theta_L} \geq \hat{\gamma}(0)(\theta_H - \theta_L).$$

   This imposes constraint

   $$\bar{a} \leq (\gamma - \hat{\gamma}(0))(\theta_H - \theta_L)\theta_L.$$

3. The good worker should not want to deviate:

$$\gamma\left(\theta_H-\theta_L\right)-\frac{\bar{a}}{\theta_H}\geq\hat{\gamma}\left(a\right)\left(\theta_H-\theta_L\right)-\frac{a}{\theta_H}\quad\forall a$$

4. In particular, the good worker should not want to deviate to $a=0$, which imposes constraint

$$\bar{a}\leq\left(\gamma-\hat{\gamma}\left(0\right)\right)\left(\theta_H-\theta_L\right)\theta_H.$$

It is easy to check that the conditions in 1 through 4 are satisfied if

$$\begin{aligned}\hat{\gamma}\left(a\right) &= \begin{cases} 0 & \text{if } a<\bar{a} \\ \gamma & \text{if } a\geq\bar{a}. \end{cases} \\ \bar{a} &\leq \gamma\left(\theta_H-\theta_L\right)\theta_L.\end{aligned}$$

which determines a continuum of pooling equilibria.

Intuitive Criterion

As we saw in the last example, there may be multiple perfect Bayesian equilibria.

Can we select among equilibria?

Intuitive Criterion: Cho and Kreps 1987

Idea: some out-of-equilibrium beliefs are unreasonable.

Suppose there are only two types: $\theta \in \{\theta_1, \theta_2\}$

Take a PBE and consider out-of-equilibrium action $\hat{a}_1$

If:

1. Type θ_1 gets a strictly higher utility by deviating to $\hat{a}_1$ if the receiver has belief $\mu(\theta = \theta_1 | \hat{a}_1) = 1$;

2. Type θ_2 does not get a strictly higher utility by deviating to $\hat{a}_1$ for any belief $\mu(\theta = \theta_1 | \hat{a}_1)$ that the receiver may hold.

Then, we say that the PBE fails the Intuitive Criterion.

Intuition: the Good Type tells the receiver:

"I am going to make a deviation that cannot possibly be profitable for a Bad Type. Therefore, when I deviate, you must believe I am a Good Type."

Applying the Intuitive Criterion to Spence:

Proposition 22 *The only PBE that survives the Intuitive Criterion is the separating equilibrium with* $a(\theta_H) = \theta_L(\theta_H - \theta_L)$.

Separating Equilibria

Take a separating equilibrium with $\theta_L(\theta_H - \theta_L) < a(\theta_H) \leq \theta_H(\theta_H - \theta_L)$ and consider a deviation to $\hat{a} = \theta_L(\theta_H - \theta_L)$.

1. Type θ_H gets a strictly higher utility by deviating to $\hat{a}$ if the receiver has belief $\hat{\gamma}(\hat{a}) = 1$, because

$$\theta_H - \frac{\theta_L(\theta_H - \theta_L)}{\theta_H} > \theta_H - \frac{a(\theta_H)}{\theta_H}.$$

2. Type θ_2 does not get a strictly higher utility by deviating to $\hat{a}_1$ for any belief $\hat{\gamma}(\hat{a}) \in [0,1]$ because

$$\begin{aligned} & \theta_L + \hat{\gamma}(\hat{a})(\theta_H - \theta_L) - \frac{\theta_L(\theta_H - \theta_L)}{\theta_L} \\ \leq \ & \theta_L \quad \forall \hat{\gamma}(\hat{a}) \in [0,1]. \end{aligned}$$

The only separating equilibrium that survives the Intuitive Criterion is the one in which

$$a(\theta_H) = \theta_L(\theta_H - \theta_L).$$

Pooling Equilibria

Take a pooling equilibrium with $a(\theta_L) = a(\theta_H) = \bar{a} \leq \gamma(\theta_H - \theta_L)\theta_H$. and consider a deviation to

$$\hat{a} = \bar{a} + (1-\gamma)(\theta_H - \theta_L)\theta_L.$$

1. Type θ_H gets a strictly higher utility by deviating to $\hat{a}$ if the receiver has belief $\hat{\gamma}(\hat{a}) = 1$, because
$$\begin{aligned} & \theta_H - \frac{\hat{a}}{\theta_H} = \theta_H - \frac{(1-\gamma)(\theta_H-\theta_L)\theta_L}{\theta_H} - \frac{\bar{a}}{\theta_H} \\ > & \ \theta_H - \frac{(1-\gamma)(\theta_H-\theta_L)\theta_L}{\theta_L} - \frac{\bar{a}}{\theta_H} \\ = & \ \gamma\theta_H + (1-\gamma)\theta_L - \frac{\bar{a}}{\theta_H}. \end{aligned}$$

2. Type θ_2 does not get a strictly higher utility by deviating to $\hat{a}_1$ for any belief $\hat{\gamma}(\hat{a}) \in [0,1]$ because
$$\theta_H - \frac{\hat{a}}{\theta_L} = \gamma\theta_H + (1-\gamma)\theta_L - \frac{\bar{a}}{\theta_L}.$$

All pooling equilibria fail the Intuitive Criterion.

Spence Signalling in Practice

Two stories about education:

- Productivity increase.
- Sorting device.

Very different policy implications.

How can you tell them apart empirically?

Weiss (1995), Altonji (1995), Altonji and Pierret (2001)

Generalization of PBE

We focused on signalling games:

- only two players;

- in each period, only one player moves.

In other cases, PBE may be too weak.

Sequential equilibrium (Kreps-Wilson 1982)

Lecture 5: Moral Hazard

Mas Colell, Whinston, Green

Principal-agent: hidden type or hidden action

Definitions

π: observed profit, $\pi \in [\pi_L, \pi_H]$.

e: agent's action (effort), $e \in \{e_L, e_H\}$.

Conditional profit density:

$$f(\pi|e) \quad \text{with } f > 0 \text{ for all } \pi, e$$

First-order stochastic dominance:

$$F(\pi|e_H) \leq F(\pi|e_L) \quad \text{for all } \pi$$

with a strict inequality for some π, implying (by integration by parts):

$$E(\pi|e_H) > E(\pi|e_L).$$

Agent maximizes $v(w) - g(e)$ with $v' > 0$, $v'' \leq 0$, and $g(e_H) > g(e_L)$.

Principal maximizes $\pi - w$

Reservation value $\bar{u}$

Observable Effort

Contract specifies e and $w(\pi)$

$$\max_{e,w} \int_{\pi_L}^{\pi_H} (\pi - w(\pi)) f(\pi|e)\, d\pi$$

subject to

$$\int_{\pi_L}^{\pi_H} v(w(\pi)) f(\pi|e)\, d\pi - g(e) \geq \bar{u}. \qquad \text{(IR)}$$

Step 1 Given e, what is the cheapest compensation scheme that implements it?

$$\min_{w} \int_{\pi_L}^{\pi_H} w(\pi) f(\pi|e)\, d\pi$$

subject to

$$\int_{\pi_L}^{\pi_H} v(w(\pi)) f(\pi|e)\, d\pi - g(e) \geq \bar{u}.$$

Lagrangian

$$\begin{aligned} L &= -\int_{\pi_L}^{\pi_H} w(\pi) f(\pi|e)\, d\pi \\ &\quad +\gamma \left(\int_{\pi_L}^{\pi_H} v(w(\pi)) f(\pi|e)\, d\pi - g(e) - \bar{u} \right) \end{aligned}$$

First-order condition

$$\forall \pi \quad f\left(\pi|e\right) = \gamma v'\left(w\left(\pi\right)\right) f\left(\pi|e\right)$$

which rewrites as

$$\forall \pi \quad \frac{1}{v'\left(w\left(\pi\right)\right)} = \gamma.$$

If $v'' > 0$ (risk aversion), $v'\left(w\left(\pi\right)\right)$ is a constant. Therefore, $w\left(\pi\right) = w_e^*$ such that $v\left(w_e^*\right) - g(e) = \bar{u}$.

If $v'' = 0$ (risk neutrality), any w that satisfies IR works.

Step 2 Choose the optimal e

$$\max_e \int_{\pi_L}^{\pi_H} \pi f\left(\pi|e\right) d\pi - v^{-1}\left(\bar{u} + g(e)\right)$$

With a risk-neutral agent ($v\left(w\right) = w$),

$$\max_e \int_{\pi_L}^{\pi_H} \pi f\left(\pi|e\right) d\pi - g(e)$$

Unobservable Effort: Risk-Neutral Agent

The optimal contract involves "selling the store"

$$w(\pi) = \pi - \bar{w}.$$

The agent solves

$$e^* = \text{arg}\max_e \int_{\pi_L}^{\pi_H} \pi f(\pi|e)\, d\pi - g(e).$$

The principal sets

$$\int_{\pi_L}^{\pi_H} \pi f(\pi|e^*)\, d\pi - g(e^*) - \bar{w} = \bar{u}.$$

Surplus maximization and zero rent.

Proposition 26 *With a risk neutral agent and unobservable effort, the optimal contract results in the same effort and the same utility as when effort is observed.*

Unobservable Effort: Risk-Averse Agent

Cost of implementing e

$$\min_{w} \int_{\pi_L}^{\pi_H} w(\pi) f(\pi|e)\, d\pi$$

subject to

$$\int_{\pi_L}^{\pi_H} v(w(\pi)) f(\pi|e)\, d\pi - g(e) \geq \bar{u}. \qquad \text{(IR)}$$

$$e \in \arg\max_{\tilde{e}} \int_{\pi_L}^{\pi_H} v(w(\pi)) f(\pi|\tilde{e})\, d\pi - g(\tilde{e}) \quad \text{(IC)}$$

If $e = e_L$, IC is easy to satisfy. Set a constant wage w that satisfies IR as an equality. IC is satisfied too because $g(e_H) > g(e_L)$:

$$w^*_{e_L} = v^{-1}(\bar{u} + g(e_L)).$$

If $e = e_H$, IC becomes

$$\begin{aligned} & \int_{\pi_L}^{\pi_H} v(w(\pi)) f(\pi|e_H)\, d\pi - g(e_H) \\ \geq \; & \int_{\pi_L}^{\pi_H} v(w(\pi)) f(\pi|e_L)\, d\pi - g(e_L) \end{aligned}$$

Lagrangian:

$$\begin{aligned}
L = & -\int_{\pi_L}^{\pi_H} w(\pi) f(\pi|e_H)\, d\pi \\
& +\gamma\left(\int_{\pi_L}^{\pi_H} v(w(\pi)) f(\pi|e_H)\, d\pi - g(e_H) - \bar{u}\right) \\
& +\mu\left(\int_{\pi_L}^{\pi_H} v(w(\pi)) f(\pi|e_H)\, d\pi - g(e_H)\right. \\
& \left. -\int_{\pi_L}^{\pi_H} v(w(\pi)) f(\pi|e_L)\, d\pi - g(e_L)\right).
\end{aligned}$$

Foc: for all π,

$$\begin{aligned}
\frac{\partial L}{\partial w} = & -f(\pi|e_H) + \gamma v'(w(\pi)) f(\pi|e_H) \\
& +\mu\left(v'(w(\pi)) f(\pi|e_H) - v'(w(\pi)) f(\pi|e_L)\right)
\end{aligned}$$

which rewrites as

$$\frac{1}{v'(w(\pi))} = \gamma + \mu\left(1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}\right) \qquad \text{(FOC)}$$

Lemma 27 *If $e = e_H$, $\gamma > 0$ and $\mu > 0$.*

Proof: Suppose $\gamma = \mu = 0$. Contradiction because $v' > 0$.

Suppose $\gamma = 0$. $\exists\tilde{\Pi}$ such that $\forall\pi \in \tilde{\Pi}$ we have that $\frac{f(\pi|e_L)}{f(\pi|e_H)} > 1$. Then, for $\pi \in \tilde{\Pi}$

$$\frac{1}{v'\left(w\left(\pi\right)\right)} = 0 + \mu\left(-\right) < 0$$

Suppose $\mu = 0$. $w\left(\pi\right)$ is constant $\Longrightarrow$ the agent chooses $e = e_L$. $\square$

Interpreting FOC

$$\frac{1}{v'(w(\pi))} = \gamma + \mu \left(1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}\right)$$

Let $\hat{w}$ be such that $\frac{1}{v'(\hat{w})} = \gamma$.

$$\begin{aligned} \frac{f(\pi|e_L)}{f(\pi|e_H)} &< 1 \Longrightarrow w(\pi) > \hat{w} \\ \frac{f(\pi|e_L)}{f(\pi|e_H)} &> 1 \Longrightarrow w(\pi) < \hat{w} \end{aligned}$$

$\frac{f(\pi|e_L)}{f(\pi|e_H)}$: likelihood ratio (but beware of a statistical interpretation).

The wage w need not be increasing in π.

Sufficient condition: monotone likelihood ratio property (Milgrom 1982).

$$\frac{f(\pi|e_L)}{f(\pi|e_H)} \text{ decreasing in } \pi$$

[FIGURES]

Effort Choice

The cost of implementing e_L is the same as under observable effort.

The cost of implementing e_H is higher because of variance and risk aversion.

Summary

With unobservable effort:

- the contract to implement e_L is flat.

- the contract to implement e_H is given by FOC and has a higher expected payment than under observable e.

- non-observability may cause a welfare loss.

Additional Signal

Suppose that the principal can observe π and an additional signal y. Should compensation depend on y as well?

Holmstrom (1979): yes, unless π is a sufficient statistic for y:

$$\tilde{h}(y|\pi, e) = h(y|\pi)$$

To see this, note that statistical sufficiency implies:

$$\tilde{f}(\pi, y|e) = \tilde{h}(y|\pi, e) f(\pi|e) = h(y|\pi) f(\pi|e)$$

Hence, the first-order condition with the additional signal is:

$$\frac{1}{v'(w(\pi))} = \gamma + \mu \left(1 - \frac{\tilde{f}(\pi, y|e_L)}{\tilde{f}(\pi, y|e_H)}\right) = \gamma + \mu \left(1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}\right.$$

If the condition above is satisfied, the foc is identical to the foc with π only.