Production Function and technology progress

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1 Introduction

- Technology progress can be seen as transformation of production function. ——Joseph Schumpeter

- Given a output level, the input combination should adjust accordingly when their prices changes.

  However, marginal rate of technical substitution is often decreasing.

- Induced innovation: When one input becomes more expensive, there will arise R&D incentive to alleviate the scarcity.

- For example, why industrial revolution happens in UK, rather than in China, even where commodity market had been very advancedly developed?

- One explanation is so-called "high-level equilibrium trap", that is, there were ample labors in China, which implies that it is "economical" to save labor by innovating something like steamers.

2 Capital-saving technological progress

- Keeping K/L constant, the marginal rate of technical substitution is increasing gradually when time elapses, that is

$$\frac{d}{dt}MRTS_{LK} = \frac{d}{dt}(-\frac{dK}{dL}) > 0 \tag{1}$$

- Geometrical explanation

- Comment: To produce a given output level under the same factor price vector, the proportion of K/L is lower after technology progress is realized than before.

3 Labor-saving technological progress

- Keeping K/L constant, the marginal rate of technical substitution is increasing gradually when time elapses, that is

$$\frac{d}{dt}MRTS_{LK} = \frac{d}{dt}(-\frac{dK}{dL}) < 0 \tag{2}$$

- Geometrical explanation

- Comment: To produce a given output level under the same factor price vector, the proportion of K/L is higher after technology progress is realized than before.

4 Neutral technological progress

- Keeping K/L constant, the marginal rate of technical substitution is increasing gradually when time elapses, that is

$$\frac{d}{dt} MRTS_{LK} = \frac{d}{dt}(-\frac{dK}{dL}) = 0 \qquad (3)$$

- Geometrical explanation:

- Comment: To produce a given output level under the same factor price vector, the proportion of K/L is constant after technology progress is realized than before.

5 Technological progress under homogeneity of degree one

- Defintion: $f(\lambda L, \lambda K, t) = \lambda f(L, K, t)$

- Define $\alpha_K = \frac{d \ln K}{d \ln f} = \frac{K f_K}{f}$ as the share of capital return to total production.

- Euler equation with linear homogeneity: $K f_K + L f_L = f$. So $\frac{L f_L}{f} = 1 - \alpha_K$ is the share of labor return ot total production.

- A technology progress is labor-saving when $\frac{d}{dt}\alpha_K > 0$.

- Proof: $\alpha_K = \frac{Kf_K}{f} = \frac{Kf_K}{Kf_K + Lf_L} = 1/\left[1 + \frac{L}{K}\frac{f_L}{f_K}\right]$.So when keeping $\frac{L}{K}$ constant, $\frac{d}{dt}\alpha_K > 0$ is equivalent to $\frac{d}{dt}\left(\frac{f_L}{f_K}\right) = \frac{d}{dt}\left(-\frac{dK}{dL}\right) < 0$. QED.

- Similarly, a technology progress is Neutral or capital-saving when $\frac{d}{dt}\alpha_K = 0$ or $\frac{d}{dt}\alpha_K > 0$.

6 Hicksian Neutrality

- A technology progress is Hicksian neutral when the marginal rate of technical substitution does not change with time if keeping K/L constant.

- Comment: $\frac{L}{K}\frac{f_L}{f_K}$ does not change as shown above.

- Proposition: Hicksian-neutral production function is like

$$f(L, K, t) = A(t) f(L, K) \tag{4}$$

- Geometrical explanation: Homogeneity of degree one, so $y = A(t)f(1, k)$, where $y = Y/L$, and $k = K/L$. For any given level of k (constant over time), the slope at time t_1 is $A(t_1)f'(1, k)$, and the slope at time t_2 is $A(t_2)f'(1, k)$. So the rate of two slopes is

$$\frac{A(t_1)f'(1,k)}{A(t_2)f'(1,k)} = \frac{A(t_1)f(1,k)}{A(t_2)f(1,k)} \tag{5}$$

So the two tagent lines must cross the same point on the horizontal axis.

7 Harrodian Neutrality

- A technology progress is Harrodian neutral if marginal product of capital f_K and capital-output rate q/K are both constant overtime.

- Comment: Let $f_K = a \neq a(t)$, $q/K = b \neq b(t)$, then $\alpha_K = \frac{K f_K}{f} = a/b$ is also constant overtime.

- Proposition: the necessary and sufficient condition for Harrodian Neutrality is that the production function looks like

$$f(L, K, t) = f(A(t)L, K) \tag{6}$$

- Geometrical explanation: Homogeneity of degree one, so $y = f(A(t), k)$, where $y = Y/L$, and $k = K/L$. For any given level of $b = y/k = Y/K$, we consider the line y=bk, which cross the production fucntion $f(A(t_1), k)$ at point A and $f(A(t_2), k)$ at point B. Then Harrodian Neutrality require that the slope of $f(A(t_1), k)$ at point A is the same as the slope of function $f(A(t_2), k)$ at point B.

8 Solowian Neutrality

- A technology progress is Solowian neutral if marginal product of capital f_L and capital-output rate q/L are both constant overtime.

- Comment: Let $f_L = a \neq a(t)$, $q/L = b \neq b(t)$, then $\alpha_K = 1 - \frac{Lf_L}{f} = 1 - a/b$ is also constant overtime.

- Proposition: the necessary and sufficient condition for Harrodian Neutrality is that the production function looks like

$$f(L, K, t) = f(L, A(t)K) \tag{7}$$

- Geometrical explanation: Homogeneity of degree one, so $y = f(1, A(t)k)$, where $y = Y/L$, and $k = K/L$. As a result, to hold y constant, $A(t)k$ must be constant too overtime. That is, $k_1 A(t_1) = k_2 A(t_2)$.

9 Decomposition of economic growth

- Suppose production function is

$$q = f\left[L, K, A(t)\right] \tag{8}$$

Then

$$\frac{dq}{q} = \frac{\partial \ln f}{\partial \ln A}\frac{dA}{A} + \alpha_L \frac{dL}{L} + \alpha_K \frac{dK}{K} \tag{9}$$

So, under Hicksian tech. progress, $\mu = \frac{dA}{A} = \frac{dq}{q} - \alpha_L \frac{dL}{L} + \alpha_K \frac{dK}{K}$

10 C-D function and neutrality

- Proposition: the necessary and sufficient condition under which a linear homogeneity production function is both Hicksian neutral and Solowian neutral or Harrodian neutral is that is is a C-D function:

$$q = A(t)L^{\alpha_L}K^{1-\alpha_L}, 0 < \alpha_L < 1, A(t) > 0 \tag{10}$$

11 CES function and neutrality

- Hicksian neutrality:

$$q = A(t)\left[\alpha_L L^{-\rho} + (1-\alpha_L)K^{-\rho}\right]^{-1/\rho}$$

$$\begin{aligned} \ln q &= \ln A(t) - \frac{1}{\rho}\ln\left[\alpha_L L^{-\rho} + (1-\alpha_L)K^{-\rho}\right] \\ &= \ln A_0 + \mu t - \frac{1}{\rho}\ln\left[\alpha_L L^{-\rho} + (1-\alpha_L)K^{-\rho}\right]^{-1/\rho} \end{aligned}$$

if we assume that $A(t) = A_0 e^{\mu t}$.

So, the rate of tech. progress is μ.

- Harrodian neutrality

$$q = B\left[\alpha_L\left[A(t)L\right]^{-\rho} + (1-\alpha_L)K^{-\rho}\right]^{-1/\rho}$$

$$\begin{aligned}\ln q &= \ln B - \frac{1}{\rho}\ln\left[\alpha_L\left[A(t)L\right]^{-\rho} + (1-\alpha_L)K^{-\rho}\right] \\ &= \ln B - \frac{1}{\rho}\ln\left[\alpha_L A_0^{-\rho} e^{-\rho\mu t} L^{-\rho} + (1-\alpha_L)K^{-\rho}\right]^{-1/\rho}\end{aligned}$$

where B is a constant. So, Harrodian neutral rate of tech. progress is

$$\frac{\partial \ln f}{\partial t} = \frac{\partial \ln q}{\partial A}\frac{dA}{dt} = \frac{A_0^{-\rho+1}\mu\alpha_L e^{-\rho\mu t}L^{-\rho}}{A_0^{-\rho}\alpha_L e^{-\rho\mu t}L^{-\rho} + (1-\alpha_L)K^{-\rho}} \tag{11}$$

similarly, we can get Solowian neutral rate. They are not constant, but change overtime.

12 Translog function and neutrality

- Hicksian

$$
\begin{aligned}
\ln q &= \alpha_0 + \ln A(t) + \alpha_L \ln L + \alpha_K \ln K + \\
&\quad \frac{1}{2}\beta_{11}\left[\ln L\right]^2 + \beta_{12} \ln L \ln K + \frac{1}{2}\beta_{22}\left[\ln K\right]^2
\end{aligned}
$$

Assuming $A(t) = A_0 e^{\mu t}$, then

$$
\begin{aligned}
\ln q &= \left[\alpha_0 + \ln A_0\right] + \mu t + \alpha_L \ln L + \alpha_K \ln K + \\
&\quad \frac{1}{2}\beta_{11}\left[\ln L\right]^2 + \beta_{12} \ln L \ln K + \frac{1}{2}\beta_{22}\left[\ln K\right]^2
\end{aligned}
$$

So, the technological rate of progress is μ.

- Harrodian

$$
\begin{aligned}
\ln q &= \alpha_0 + \alpha_L \ln [A(t)L] + \alpha_K \ln K + \\
&\frac{1}{2}\beta_{11} [A(t)L]^2 + \beta_{12} \ln [A(t)L] \ln K + \frac{1}{2}\beta_{22} [\ln K]^2
\end{aligned}
$$

Assuming $A(t) = A_0 e^{\mu t}$, then

$$\begin{aligned}
\ln q &= [\alpha_0 + \alpha_L \ln A_0] + \frac{1}{2}\beta_{11}[\ln A_0]^2 \\
&\quad + [\alpha_L \mu + \mu\beta_{11}\ln(A_0)]t + \frac{1}{2}\beta_{11}\mu^2 t^2 \\
&\quad + [\alpha_L + \beta_{11}\ln(A_0) + \beta_{11}\mu t]\ln L \\
&\quad + [\alpha K + \beta_{12}\ln(A_0) + \beta_{12}\mu t]\ln K \\
&\quad + \frac{1}{2}\beta_{11}[\ln L]^2 + \beta_{12}\ln L \ln K \\
&\quad + \frac{1}{2}\beta_{22}[\ln K]^2
\end{aligned}$$

so, the Harodian neutral technological rate of progress is

$$\frac{\partial \ln f}{\partial t} = [\alpha_L + \beta_{11} \ln(A_0)]\mu + \beta_{11}\mu^2 + [\beta_{11} \ln L + \beta_{12} \ln K]\mu \tag{12}$$

- Similarly, we can obtain the Solowian rate of tech. progress.

13 TFP(Total Factor Productivity)

- Algebra Index

$$LP_t = \frac{\frac{q_t}{q_0}}{\alpha_{L_0}\frac{L_t}{L_0} + \alpha_{K_0}\frac{K_t}{K_0}} \tag{13}$$

where LP_t is the factor productivity index in year t. α_{L_0} is the labor production elasticity at base year with $\alpha_{L_0} = \left(\frac{\partial q}{\partial L} / \frac{q}{L}\right)_{t=t_0}$; α_{K_0} is capital production elasticity at base year with $\alpha_{K_0} = \left(\frac{\partial q}{\partial K} / \frac{q}{K}\right)_{t=t_0}$, q_0, L_0, K_0 are the production, labor and capital respectively in base year. t_0 is the base year and q_t is the production in year t.

If factor markets are perfectly competitive and the wage and interest rate in base year are w_0 and r_0 respectively, then

$$w_0 = \left(\frac{\partial q}{\partial L}\right)_{t=t_0}, r_0 = \left(\frac{\partial q}{\partial K}\right)_{t=t_0} \tag{14}$$

$$\alpha_{L_0} = \frac{w_0 L_0}{q_0}, \alpha_{K_0} = \frac{r_0 K_0}{q_0} \tag{15}$$

then

$$LP_t = \frac{q_t}{w_0 L_t + r_0 K_t} \tag{16}$$

Geometric Index

$$
\begin{aligned}
LP_t &= \frac{\frac{q_t}{q_0}}{(\frac{L_t}{L_0})^{\alpha_L} + (\frac{K_t}{K_0})^{\alpha_K}} \\
&= \frac{\frac{q_t}{(L_t)^{\alpha_L}(K_t)^{\alpha_K}}}{\frac{q_0}{(L_0)^{\alpha_L}(K_0)^{\alpha_K}}} \qquad (17)
\end{aligned}
$$

If production function is C-D, then

$$
LP_t = \frac{A_t}{A_0} \qquad (18)
$$