The Theory of the Firm

Zonglai KOU
Fudan University

May, 2007

1 Introduction

- Black-box approach to the firm in neoclassical economics

The hierarchy within firm is neglected to focus on the inter-firm competitions. However, coporate governance is very important because either adverse selection or moral hazard problem will arise when there is informational asymmetry between Principal and Agent.

One interesting direction is to discuss the effect of corporate governance on the performance at industrial level.

- A firm is a technical unit in which commodities are produced. When there is only one output, we can define a production function; when there are more than one outputs, we can define a correspondence.

- Short-term or long-term: whether one or more than one inputs are invariant in the period?

- The similarity between the theory of consumption and the theory of the firm

A consumer purchases commodities wiht which he "produces" satisfaction; An entrepreneur purchases inputs with which he produces commodities.

The consumer's budget equation is a linear function of the amounts of commodities he purchases; the competitive firm's cost equation is a linear function of the amounts of inputs it purchases.

- The difference between the theory of consumption and the theory of the firm

Utility function is subjective; Production function is objective. The rational consumer maximizes utility for a given income; but the entrepreneur often considers his cost variable.

2 Production Set

A production vector is a vector $y = (y_1, y_2, ...y_L) \in R^L$ that describes the (net) outputs of the L commodities from a production process.

- Example: Suppose L=5. Then y=(-5,2.-6,3,0) is a production vector.

The set of all production vectors that constitute feasible plans for the firm is known as the production set and is denoted by $Y \in R^L$, any $y \in Y$ is possible, and any $y \notin Y$ is not.

- Properties of the production sets

1. Y is nonempty

2. Y is closed. The set Y includes its boundary. $y^n \to y$, and $y^n \in Y$ implies $y \in Y$.

3. No free lunch. At least one term in y is negative.

4. Possibility of inaction. That is, 0 $\in Y$. But this is not the case when there is Sunk Cost.

5. Free disposal. $Y - R^L_+ \subset Y$.

6. Irreversibility. Suppose that $y \in Y$, and $y \neq$ 0. Then, $-y \notin Y$.

7. Nonincreasing returns to scale: if for any $y \in Y$, we have $\alpha y \in Y$ for all $\alpha \in [0, 1]$.

8. Nondecreasing returns to scale: if for any $y \in Y$, we have $\alpha y \in Y$ for all $\alpha \geq 1$.

9. Nondecreasing returns to scale: if for any $y \in Y$, we have $\alpha y \in Y$ for any $\alpha > 0$.

10. Additivity (or free entry). Suppose that $y \in Y$ and $y \in Y$, then additivity requires taht $y + y' \in Y$.

11. Convexity. This one of the fundamental assumptions of microeconomics. That is, if $y, y' \in Y$ and $\alpha \in [0, 1]$, then $\alpha y + (1 - \alpha)y' \in Y$.

3 The Production Function

$$q = f(x_1, x_2) \tag{1}$$

where (1) is assumed to be a single-valued contrinuous functin with continuous first- and second-order derivatives; $f_i > 0$, $f_{ii} < 0$, $f_{ij} > 0$ in most cases.

Remark 1: The production function differs from the technolgoy in that it presupposes technoical efficiency and states the maximum output abtainable from every possible input combination.

Remark 2: The best utilization of any particular input combination is a technical, not an economic, problem.

- Product Curves

By fixing factor $x_2 = x_2^0$, we obtain the relationship between q and x_1 :

$$q = f(x_1, x_2^0) \tag{2}$$

- Average product

$$AP = \frac{q}{x_1} = \frac{f(x_1, x_2^0)}{x_1} \tag{3}$$

- Marginal product

$$MP = \frac{\partial q}{\partial x_1} = f_1(x_1, x_2^0) \tag{4}$$

- The Output Elasticity of X_1

$$\omega_1 = \frac{\partial(\ln q)}{\partial(\ln x_1)} = \frac{x_1 \partial q}{q \partial x_1} = \frac{MP}{AP} \tag{5}$$

- Isoquants

An isoquant is the locus of all cominations of x_1 and x_2 which yield a specified output level. For a given output level, (1) becomes

$$q_0 = f(x_1, x_2) \tag{6}$$

where q_0 is a parameter.

- The rate of technical substitution (RTS)

$$RTS = -\frac{dx_2}{dx_1} \tag{7}$$

- Economic sense: the slope of the tangent to a point on an isoquant is the rate at which x_1 msut be substituted for x_2 in order to maintain the corresponding output level. Totally differentiating the production function leads to

$$dq = f_1 dx_1 + f_2 dx_2 = 0 \tag{8}$$

where the last equality is satisfied when $q = q_0$. As a result, we obtain

$$RTS = -\frac{dx_2}{dx_1} = \frac{f_1}{f_2} \tag{9}$$

that is, the RTS at a point equals the ratio of the MP of x_1 to the MP x_2 at that point.

- Excercise 1: Derive the RTS of Cobb-Douglas function $q = f(x_1, x_2) = x_1^{\alpha}, x_2^{1-\alpha}$.

- Elasticity of Substitition

Elasticity of Substitition σ is a pure number that measures the rate at wihch substitution takes palce. It is defined as the proportionate rate of change of the inout ratio divided by the proportionate rate of change of the RTS

$$\sigma = \frac{\partial \ln(x_2/x_1)}{\partial \ln(f_1/f_2)} = \frac{f_1/f_2}{x_2/x_1}\frac{d(x_2/x_1)}{d(f_1/f_2)} \tag{10}$$

- Excercise 2: Prove that the class of production functions given by $q = Ax_1^{\alpha}x_2^{\beta}$ with $\alpha, \beta > 0$ has unit elasiticity of substitution, that is, $\sigma = 1$.

- Excercise 3: Derive the elasticity of substitution of production function $q = B[\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho}]^{-1/\rho}$ with $\rho > -1$ and explain the economic meaning of parameter ρ. Hints: What is the relationship between ρ and σ, and what happens when ρ increases from -1 to infinite?

4 Optimizing Behavior

The entrepreneur purchases x_1 and x_2 in perfectly competitive market at constrant unit prices. His total cost of prouction (C) is given by the linear equation

$$C = r_1 x_1 + r_2 x_2 + b \tag{11}$$

- Constrained Output Maximization: the entrepreneur maximizes his output subject to cost constraint that $C \leq C^0$:

$$Lagrangian = V = f(x_1, x_2) + \mu(C^0 - r_1 x_1 - r_2 x_2 - b) \tag{12}$$

- First order condtions (FOC)

$$\frac{\partial V}{\partial x_1} = f_1 - \mu r_1 = 0 \quad (13)$$

$$\frac{\partial V}{\partial x_2} = f_2 - \mu r_2 = 0 \quad (14)$$

$$\frac{\partial V}{\partial \mu} = C^0 - r_1 x_1 - r_2 x_2 - b = 0 \quad (15)$$

As a result, the ratio of the MPs of x_1 and x_2 must be equated with the ratio of their prices

$$\frac{f_1}{f_2} = \frac{r_1}{r_2} \quad (16)$$

and the contribution to output of the last dollar expended upon each input must equal μ,

$$\mu = \frac{f_1}{r_1} = \frac{f_2}{r_2} \tag{17}$$

Furthermore,

$$RTS = \frac{r_1}{r_2} \tag{18}$$

The second-order conditions rquire that the relevant bordered Hessian determinant be positive

$$\begin{vmatrix} f_{11} & f_{12} & -r_1 \\ f_{21} & f_{22} & -r_2 \\ -r_1 & -r_2 & 0 \end{vmatrix} > 0 \tag{19}$$

- Constrained Cost Minimization

$$Lagrangian = Z = r_1 x_1 + r_2 x_2 + \lambda f(x_1, x_2) \tag{20}$$

$$\frac{\partial Z}{\partial x_1} = r_1 - \lambda f_1 = 0 \tag{21}$$

$$\frac{\partial V}{\partial x_2} = r_2 - \lambda f_2 = 0 \tag{22}$$

$$\frac{\partial V}{\partial \mu} = q^0 - f(x_1, x_2) = 0 \tag{23}$$

$r_1x_1 + r_2x_2 + b$Similarly, we obtain

$$\frac{f_1}{f_2} = \frac{r_1}{r_2} \text{ or } \frac{1}{\lambda} = \frac{f_1}{r_1} = \frac{f_2}{r_2} \text{ or } RTS = \frac{r_1}{r_2} \tag{24}$$

Now the second order condition requires taht the relevant bordered Hessian determinant be negative

$$\begin{vmatrix} -\lambda f_{11} & -\lambda f_{12} & -f_1 \\ -\lambda f_{21} & -\lambda f_{22} & -f_2 \\ -f_1 & -f_2 & 0 \end{vmatrix} < 0 \tag{25}$$

- Excercise 4: Prove that the SOC (19) is equivalent to the SOC (25).

If the production function is regular strictly quasi-concave, every point of tangency between an isoquant and an isocost line is the solution of both a constrainted-maximum and costrained-minimum problem.

- Expansion Path, i.e., the locus of tangency points, is defined by an implicit function of x_1 and x_2

$$g(x_1, x_2) = 0 \tag{26}$$

- Excercise 5: Derive the expansion path of a Cobb-Douglas function.

- Profit maximization: Direct approach

Suppose that product price is exogenously given (what does it mean?), then the profit maximization problem is

$$\max_{x_1, x_2} \pi \; pf(x_1, x_2) - (r_1 x_1 + r_2 x_2 + b) \tag{27}$$

FOCs

$$\begin{aligned} \frac{\partial \pi}{\partial x_1} &= pf_1 - r_1 = 0 \\ \frac{\partial \pi}{\partial x_2} &= pf_2 - r_2 = 0 \\ \frac{f_1}{r_1} &= \frac{f_2}{r_2} \end{aligned} \tag{28}$$

Second order condtions require that the principal minors of the relevant Hessian determinant alternate in sign:

$$\frac{\partial^2 \pi}{\partial x_1^2} = pf_{11} < 0; \frac{\partial^2 \pi}{\partial x_2^2} = pf_{22} < 0 \tag{29}$$

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1^2} & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi}{\partial x_2^2} \end{vmatrix} = p^2 \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0 \tag{30}$$

SOC (29) and (30) reqire that the production function be strictly concave in the neighborhood of a point at which the first-order are satisified with $x_1, x_2 \geq 0$ if such a point exists.

5 Input Demands

- Input Demand Functions

The producer's input demands are derived from the underlying demand for the commodity which he produces. His input demand functions are obatined by sovling his firt-order conditons (28) for x_1 and x_2 as functions of r x_1, x_2 and p.

Consider production function $q = Ax_1^{\alpha}x_2^{\beta}$ with $\alpha, \beta > 0$ and $\alpha + \beta < 1$.

$$\pi = pAx_1^{\alpha}x_2^{\beta} - (r_1x_1 + r_2x_2) \tag{31}$$

The first order conditions are as follows:

$$\frac{\partial \pi}{\partial x_1} = p\alpha A x_1^{\alpha-1} x_2^{\beta} - r_1 = 0 \qquad (32)$$

$$\frac{\partial \pi}{\partial x_2} = p\beta A x_1^{\alpha} x_2^{\beta-1} - r_2 = 0$$

- Excercise 6: Prove that the input demand functions in this case are as follows:

$$x_1 = (\frac{\alpha}{r_1})^{(1-\beta)/\gamma}(\frac{\beta}{r_2})^{\beta/\gamma}(Ap)^{1/\gamma} \qquad (33)$$

$$x_2 = (\frac{\alpha}{r_1})^{\alpha/\gamma}(\frac{\beta}{r_2})^{(1-\alpha)/\gamma}(Ap)^{1/\gamma} \qquad (34)$$

where $\gamma = 1 - \alpha - \beta$.

- Comparative statics analysis

How the input demands change with product price p and input prices r_1 and r_2?

Differentiating (32) totally and reagrranging terms,

$$pf_{11}dx_1 + pf_{12}dx_2 = -f_{1dp} + dr_1 \quad (35)$$

$$pf_{21}dx_1 + pf_{22}dx_2 = -f_{2dp} + dr_2 \quad (36)$$

Solving (35) for dx_1 and dx_2 by Cramer's rule,

$$\begin{aligned} dx_1 &= \frac{1}{pH}\left[f_{22}dr_1 - f_{12}dr_2 + (f_{12}f_2 - f_{22}f_1)\,dp\right] \\ dx_2 &= \frac{1}{pH}\left[-f_{21}dr_1 + f_{11}dr_2 + (f_{21}f_1 - f_{11}f_2)\,dp\right] \end{aligned} \tag{37}$$

where

$$H = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} > 0 \tag{38}$$

Dividing both sides of the first equation of (37) by dr_1 and letting $dr_2 = dp = 0$,

$$\frac{\partial x_1}{\partial r_1} = \frac{f_{22}}{pH} < 0 \tag{39}$$

Remark: *Ceteris paribus*, the rate of change of the producer's purchase of x_1 with respect to changes inits own price is always negative, and the producer's input demand curves are always downward sloping. Here there is only a substitution effect. There is no counterpart for the income effect of the consumer in the theory of the profit-maximizing producer because he does not face a cost budget!

Dividing both sides of the first equation of (37) by dr_2 and letting $dr_1 = dp = 0$,

$$\frac{\partial x_1}{\partial r_2} = -\frac{f_{12}}{pH} \tag{40}$$

In usual cases, $f_{12} > 0$. Therefore, an increase in one input prie normally will reduce the suage the other input

Dividing both sides of the first equation of (37) by dp and letting $dr_1 = dr_2 = 0$,

$$\frac{\partial x_1}{\partial p} = -\frac{(f_{12}f_2 - f_{22}f_1)}{pH} \tag{41}$$

- An application of the Le Chaterlier Principle

The profit function for the n-input case is

$$\max \pi = f(x_1, x_2) - \sum_{i=1}^{n} r_i x_i \tag{42}$$

The Le Chaterlier Principle states that

$$\left(\frac{\partial x_i^*}{\partial r_i}\right)_0 \leq \left(\frac{\partial x_i^*}{\partial r_i}\right)_1 \leq ... \leq \left(\frac{\partial x_i^*}{\partial r_i}\right)_{n-1}, i = 1, ..., n \tag{43}$$

where the subscript outside the parenthteses desginates that the number of additional constgraints that have been appenned to the maximizaiton of (42).

The subscript 0 denotes unconstrained optimiazation, 1 denotes a case in which (42) is maximized subject to one constraint, and so on. The constraints are constructed so that x_i^* are optimal regardless of the number of constraints.

The abosolute value of demand reduction following a price increase cannot be increased as additional cosntraints are indtroduced, and may be decreased.

Intuitive explantion: $f_{ij} \geq 0$.

- Excercise 7: Prove Le Chatelier Principle with $n = 2$.

6 Cost Minimization

$$\begin{aligned} Min \;\; r.x &= \sum_{i=1}^{n} r_i x_i && (44) \\ \text{s.t. } q &\leq f(x_1, x_2, ..., x_n) = f(x) \\ x &\geq 0 && (45) \end{aligned}$$

$$Lagrangian = -r.x + \zeta(f(x) - q) \quad (46)$$

First-order conditions

$$\begin{aligned} \zeta f_i(x)) - r_i &\leq 0, && (47) \\ (\zeta f_i(x)) - r_i).x_i &= 0 && (48) \\ q &= f(x) && (49) \end{aligned}$$

As a result

$$\begin{aligned} x_i^* &= x_i^*(r_1, r_2, ..., r_n, q) && (50) \\ x^* &= x^*(r, q) && (51) \end{aligned}$$

$$\Phi(q, r) = \sum_{i=1}^{n} r_i x_i^* + \underbrace{\zeta(q - f(x_1^*, x_2^*, ..., x_n^*))}_{=0} \qquad (52)$$

- The properties of $\Phi(q, r)$

1. $\Phi(q, r)$ is homeogeneous of degree one in r and nondecreasing in q.

   Proof: If one or more input prices increase and those inputs are used at positive levels, it is necessary to move to a higher iso-cost line to secure any specified output.

2. Concavity: economic sense!

   Proof: We prove a special case with $n = 2$. For a specified output let $(r_1^0, r_2^0, x_1^0, x_2^0)$ and $(r_1^1, r_2^1, x_1^1, x_2^1)$ denote two cost-minimizing solutions. Let $r_i^2 = \lambda r_i^0 + (1 - \lambda) r_i^1 (i = 1, 2)$. By cost minimization

$$r_1^0 x_1^2 + r_2^0 x_2^2 \geq \Phi(q, r_1^0, r_2^0) = r_1^0 x_1^0 + r_2^0 x_2^0 \tag{53}$$

$$r_1^1 x_1^2 + r_2^1 x_2^2 \geq \Phi(q, r_1^1, r_2^1) = r_1^1 x_1^1 + r_2^1 x_2^1 \tag{54}$$

Consequently, (53) times λ plus (54) times λ leads to

$\Phi(q, r_1^2, r_2^2) \geq \lambda\Phi(q, r_1^0, r_2^0) + (1 - \lambda)\Phi(q, r_1^1, r_2^1)$.

$Q.E.D$

3. If the sets $\{x \geq 0 : f(x) \geq q\}$ are convex for every q, then $Y = \{(-x, q) : r.x \geq \Phi(r, q)$ for all $r >> 0\}$

4. $x^*(.)$ is homogeneous of degree zero in r.

5. Shepherd Lemma: $x_i^* = \frac{\partial}{\partial r_i}\Phi(q, r_1, r_2, ...)$

   Proof: By (52),

$$\begin{aligned}\frac{\partial \Phi(.,..)}{\partial r_i} &= x_i^*(.) + \sum_{k=1}^{n} \underbrace{(r_k - \zeta f_k(x_1^*, x_2^*, ..., x_n^*))}_{=0} \frac{\partial x_k^*}{\partial r_i} \\ &= x_i^*(.)\end{aligned} \tag{55}$$

6. If $f(.)$ is homogeneous of degree one (i.e., exhibits constant returns to scale), then $\Phi(.)$ and $x^*(.)$ are homogenous of degree one in q.

7. If $f(.)$ is concave, then $\Phi(.)$ is a convex function ofof q. in particular, marginal ocsts are nondecreasing in q.

7 Profit maximization

Price taking assumption, and $p >> 0$.

$$\pi(p) = \max p.y \quad \text{s.t.} \quad y \in Y \quad \text{or } F(y) \leq 0 \tag{56}$$

$$p_l = \lambda \frac{\partial F(y^*)}{\partial y_l} \text{ for } l = 1, ...L \tag{57}$$

or

$$p = \lambda \nabla F(y^*) \tag{58}$$

- The properties of cost function $\pi(p)$

- homogeneous of degree one in price

- convex: what is the economic sense?

- If Y is convex, then $Y = \{y \in R^L : p.y \leq \pi(p)$ for all $p >> 0$.

  Remark: If Y is closed, convex and satisfies free disposal, then $\pi(p)$ provides an alternative (dual) description of the technology.

- $y(p)$ is homogenous of degree zero.

- Hotelling Lemma: suppose $y(p)$ consists of a single point, then $\pi(p)$ is differentiable at p and $y(p) = \nabla\pi(y^*)$.

- If is a function differentiable at p, then $Dy(p) = D^2\pi(p)$ is a symmetric and positive semidefinite matrix with $Dy(p)p = 0$.

- Excercise 8: Prove the last property.

- Excercise 9: Prove $(p - p\prime).(y - y\prime) \geq 0$, where y or $y\prime$ is the profit-maxizing quantity when price is p or $p\prime$.

8 Aggregation

- Aggregation Proposition:

For all $p >> 0$, we have

(i) $\pi^*(p) = \sum_j \pi_j(p)$

(ii) $y^*(p) = \sum_j y_j(p)$ ($= \sum_j y_j : y_j \in y_j(p)$ for every j).

Economic interpretation: To find the solution of the aggregate profit maximization problem for given prices p, it is enough to add the solutions of the corresponding individual problems.