Static (or Simultaneous-Move) Games of Incomplete Information-Lecture 4

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Outline of Static Games of Incomplete Information

- *Introduction to static games of incomplete information*
- Normal-form (or strategic-form) representation of static Bayesian games
- Bayesian Nash equilibrium
- Auction

Today's Agenda

- What is a static game of incomplete information?
- Prisoners' dilemma of incomplete information
- Cournot duopoly model of incomplete information
- Battle of sexes of incomplete information
- First-price auction

Static (or simultaneous-move) games of complete information

- A set of players (at least two players)
- For each player, a set of strategies/actions
- Payoffs received by each player for the combinations of the strategies, or for each player, preferences over the combinations of the strategies

- All these are *common knowledge* among all the players.

Static (or simultaneous-move) games of INCOMPLETE information

- Payoffs are no longer common knowledge

- Incomplete information means that
  - At least one player is uncertain about some other player's payoff function (type)

- Static games of incomplete information are also called *static Bayesian games*

Prisoners' dilemma of complete information

- Two suspects held in separate cells are charged with a major crime. However, there is not enough evidence.
- Both suspects are told the following policy:
  - If neither confesses then both will be convicted of a minor offense and sentenced to one month in jail.
  - If both confess then both will be sentenced to jail for six months.
  - If one confesses but the other does not, then the confessor will be released but the other will be sentenced to jail for nine months.

		Prisoner 2	
		Mum	Confess
Prisoner 1	Mum	-1 , -1	-9 , <u>0</u>
	Confess	<u>0</u> , -9	<u>-6</u> , <u>-6</u>

Prisoners' dilemma of incomplete information

- Prisoner 1 is always rational (selfish).
- Prisoner 2 can be rational (selfish) or altruistic, depending on whether he is happy or not.
- If he is altruistic then he prefers to mum and he thinks that "confess" is equivalent to additional "four months in jail".
- Prisoner 1 can not know exactly whether prisoner 2 is rational or altruistic, but he believes that prisoner 2 is rational with probability 0.8, and altruistic with probability 0.2.

Payoffs if prisoner 2 is altruistic

		Prisoner 2	
		Mum	Confess
Prisoner 1	Mum	-1 , -1	-9 , -4
	Confess	0 , -9	-6 , -10

Prisoners’ dilemma of incomplete information cont’d

- Given prisoner 1’s belief on prisoner 2, what strategy should prison 1 choose?
- What strategy should prisoner 2 choose if he is rational or altruistic?

Payoffs if prisoner 2 is <ins>rational</ins>

		Prisoner 2: Mum	Prisoner 2: Confess
Prisoner 1	Mum	-1 , -1	-9 , $\underline{0}$
	Confess	$\underline{0}$, -9	$\underline{-6}$, $\underline{-6}$

Payoffs if prisoner 2 is <ins>altruistic</ins>

		Prisoner 2: Mum	Prisoner 2: Confess
Prisoner 1	Mum	-1 , $\underline{-1}$	-9 , -4
	Confess	$\underline{0}$, $\underline{-9}$	$\underline{-6}$, -10

Prisoners' dilemma of incomplete information cont'd

- Solution:
  - Prisoner 1 chooses to confess, given his belief on prisoner 2
  - Prisoner 2 chooses to confess if he is rational, and mum if he is altruistic
- This can be written as
  (Confess, (Confess if rational, Mum if altruistic))
- Confess is prisoner 1's best response to prisoner 2's choice (Confess if rational, Mum if altruistic).
- (Confess if rational, Mum if altruistic) is prisoner 2's best response to prisoner 1's Confess
- A Nash equilibrium called Bayesian Nash equilibrium

Cournot duopoly model of complete information

- The normal-form representation:
  - Set of players: **{ Firm 1, Firm 2}**
  - Sets of strategies: $S_1=[0, +\infty), S_2=[0, +\infty)$
  - Payoff functions:

    $u_1(q_1, q_2)=q_1(a-(q_1+q_2)-c),$

    $u_2(q_1, q_2)=q_2(a-(q_1+q_2)-c)$

- All these information is common knowledge

Cournot duopoly model of incomplete information

- A homogeneous product is produced by only two firms: firm 1 and firm 2. The quantities are denoted by q_1 and q_2, respectively.
- They choose their quantities simultaneously.
- The market price: $P(Q)=a-Q$, where a is a constant number and $Q=q_1+q_2$.
- Firm 1's cost function: $C_1(q_1)=cq_1$.
- All the above are common knowledge

Cournot duopoly model of incomplete information cont'd

- Firm 2's marginal cost depends on some factor (e.g. technology) that only firm 2 knows. Its marginal cost can be
  - HIGH: cost function: $C_2(q_2)=c_H q_2$.
  - LOW: cost function: $C_2(q_2)=c_L q_2$.
- Before production, firm 2 can observe the factor and know exactly which level of marginal cost is in.
- However, firm 1 cannot know exactly firm 2's cost. Equivalently, it is uncertain about firm 2's payoff.
- Firm 1 believes that firm 2's cost function is
  - $C_2(q_2)=c_H q_2$ with probability θ, and
  - $C_2(q_2)=c_L q_2$ with probability $1-\theta$.
- All the above are common knowledge
- The *Harsanyi Transformation*
  - Independent types

Cournot duopoly model of incomplete information cont'd

A solution for the Cournot duopoly model of incomplete information

Firm 2 knows exactly its marginal cost is high or low.

- If its marginal cost is high, i.e. $C_2(q_2) = c_H q_2$, then, for any given q_1, it will solve

$$\begin{aligned} &Max \qquad q_2[a-(q_1+q_2)-c_H] \\ &s.t. \qquad\quad q_2 \geq 0 \end{aligned}$$

- FOC: $a - q_1 - 2q_2 - c_H = 0 \;\Rightarrow\; q_2(c_H) = \frac{1}{2}(a - q_1 - c_H)$
- $q_2(c_H)$ is firm 2's best response to q_1, if its marginal cost is high.

Cournot duopoly model of incomplete information cont'd

Firm 2 knows exactly its marginal cost is high or low.

- If its marginal cost is low, i.e. $C_2(q_2) = c_L q_2$, then, for any given q_1, it will solve

$$\begin{array}{ll} Max & q_2[a-(q_1+q_2)-c_L] \\ s.t. & q_2 \ge 0 \end{array}$$

- FOC: $a - q_1 - 2q_2 - c_L = 0 \;\Rightarrow\; q_2(c_L) = \frac{1}{2}(a - q_1 - c_L)$

- $q_2(c_L)$ is firm 2's best response to q_1, if its marginal cost is low.

Cournot duopoly model of incomplete information cont'd

- Firm 1 knows exactly its cost function $C_1(q_1) = cq_1$.
- Firm 1 does not know exactly firm 2's marginal cost is high or low.
- But it believes that firm 2's cost function is $C_2(q_2) = c_H q_2$ with probability θ, and $C_2(q_2) = c_L q_2$ with probability $1-\theta$
- Equivalently, it knows that the probability that firm 2's quantity is $q_2(c_H)$ is θ, and the probability that firm 2's quantity is $q_2(c_L)$ is $1-\theta$. So it solves

$$
\begin{aligned}
Max \quad & \theta \times q_1[a-(q_1+q_2(c_H))-c] \\
& +(1-\theta)\times q_1[a-(q_1+q_2(c_L))-c] \\
s.t. \quad & q_1 \geq 0
\end{aligned}
$$

Cournot duopoly model of incomplete information cont'd

- Firm 1's problem:

$$\begin{aligned} Max \quad & \theta \times q_1[a-(q_1+q_2(c_H))-c] \\ & +(1-\theta)\times q_1[a-(q_1+q_2(c_L))-c] \\ s.t. \quad & q_1 \geq 0 \end{aligned}$$

- FOC:

$$\theta[a-2q_1-q_2(c_H)-c]+(1-\theta)[a-2q_1-q_2(c_L)-c]=0$$

Hence, $q_1 = \dfrac{\theta[a-q_2(c_H)-c]+(1-\theta)[a-q_2(c_L)-c]}{2}$

- q_1 is firm 1's best response to the belief that firm 2 chooses $q_2(c_H)$ with probability θ, and $q_2(c_L)$ with probability $1-\theta$

Cournot duopoly model of incomplete information cont'd

- Now we have

$$q_2(c_H) = \frac{1}{2}(a - q_1 - c_H)$$

$$q_2(c_L) = \frac{1}{2}(a - q_1 - c_L)$$

$$q_1 = \frac{\theta[a - q_2(c_H) - c] + (1-\theta)[a - q_2(c_L) - c]}{2}$$

- We have three equations and three unknowns. Solving these gives us

Cournot duopoly model of incomplete information cont'd

$$q_2^*(c_H) = \frac{1}{3}(a - 2c_H + c) + \frac{1-\theta}{6}(c_H - c_L)$$

$$q_2^*(c_L) = \frac{1}{3}(a - 2c_L + c) + \frac{\theta}{6}(c_H - c_L)$$

$$q_1^* = \frac{a - 2c + \theta c_H + (1-\theta)c_L}{3}$$

- Firm 1 chooses q_1^*
- Firm 2 chooses $q_2^*(c_H)$ if its marginal cost is high, or $q_2^*(c_L)$ if its marginal cost is low.
- This can be written as $(q_1^*, (q_2^*(c_H), q_2^*(c_L)))$
- One is the best response to the other
- A Nash equilibrium solution called *Bayesian Nash equilibrium*.

Cournot duopoly model of incomplete information (version one) cont'd

$$q_2^*(c_H) = \frac{1}{3}(a - 2c_H + c) + \frac{1-\theta}{6}(c_H - c_L)$$

$$q_2^*(c_L) = \frac{1}{3}(a - 2c_L + c) + \frac{\theta}{6}(c_H - c_L)$$

$$q_1^* = \frac{a - 2c + \theta c_H + (1-\theta)c_L}{3}$$

- This can be written as (q_1^* , ($q_2^*(c_H)$, $q_2^*(c_L)$))
- Firm 1 chooses q_1^* which is its best response to firm 2's ($q_2^*(c_H)$, $q_2^*(c_L)$) (and the probability).
- If firm 2's marginal cost is HIGH then firm 2 chooses $q_2^*(c_H)$ which is its best response to firm 1's q_1^* .
- If firm 2's marginal cost is LOW then firm 2 chooses $q_2^*(c_L)$ which is its best response to firm 1's q_1^* .
- **A Nash equilibrium solution called *Bayesian Nash equilibrium*.**

Cournot duopoly model of incomplete information (version two)

- A homogeneous product is produced by only two firms: firm 1 and firm 2. The quantities are denoted by q_1 and q_2, respectively.
- They choose their quantities simultaneously.
- The market price: $P(Q)=a-Q$, where a is a constant number and $Q=q_1+q_2$.

- All the above are common knowledge

Cournot duopoly model of incomplete information (version two) cont'd

- Firm 2's marginal cost depends on some factor (e.g. technology) that only firm 2 knows. Its marginal cost can be
  - HIGH: cost function: $C_2(q_2)=c_H q_2$.
  - LOW: cost function: $C_2(q_2)=c_L q_2$.
- Before production, firm 2 can observe the factor and know exactly its marginal cost is high or low.
- However, firm 1 cannot know exactly firm 2's cost. Equivalently, it is uncertain about firm 2's payoff.
- Firm 1 believes that firm 2's cost function is
  - $C_2(q_2)=c_H q_2$ with probability θ, and
  - $C_2(q_2)=c_L q_2$ with probability $1-\theta$.

Cournot duopoly model of incomplete information (version two) cont'd

- Firm 1's marginal cost also depends on some other independent factor that only firm 1 knows. Its marginal cost can be
  - HIGH: cost function: $C_1(q_1)=c_H q_1$.
  - LOW: cost function: $C_1(q_1)=c_L q_1$.
- Before production, firm 1 can observe the factor and know exactly its marginal cost is high or low.
- However, firm 2 cannot know exactly firm 1's cost. Equivalently, it is uncertain about firm 1's payoff.
- Firm 2 believes that firm 1's cost function is
  - $C_1(q_1)=c_H q_1$ with probability θ, and
  - $C_1(q_1)=c_L q_1$ with probability $1-\theta$.

Cournot duopoly model of incomplete information (version two) cont'd

- Firm 1 knows exactly its marginal cost is high or low before production.
- Firm 1 does not know exactly firm 2's marginal cost is high or low.
- But it believes that firm 2's cost function is
  - $C_2(q_2) = c_H q_2$ (in which it chooses $q_2(c_H)$) with probability θ.
  - $C_2(q_2) = c_L q_2$ (in which it chooses $q_2(c_L)$) with probability $1-\theta$.
- Now we solve firm 1's problem, given its belief on firm 2.

Cournot duopoly model of incomplete information (version two) cont'd

Firm 1 knows exactly its marginal cost is high or low.

- If its marginal cost is **HIGH**, i.e. $C_1(q_1) = c_H q_1$, then, given its belief on firm 2, it will solve

$$\begin{aligned} &Max \quad \theta \times q_1[a-(q_1+q_2(c_H))-c_H] + (1-\theta)\times q_1[a-(q_1+q_2(c_L))-c_H] \\ &s.t. \quad\ \ q_1 \geq 0 \end{aligned}$$

- FOC:

$$\theta[a-2q_1-q_2(c_H)-c_H]+(1-\theta)[a-2q_1-q_2(c_L)-c_H]=0$$

Hence, $q_1(c_H)=\dfrac{\theta[a-q_2(c_H)-c_H]+(1-\theta)[a-q_2(c_L)-c_H]}{2}$

- $q_1(c_H)$ is firm 1's best response to the belief that firm 2 chooses $q_2(c_H)$ with probability θ, and $q_2(c_L)$ with probability $1-\theta$, if firm 1's marginal cost is **HIGH**.

Cournot duopoly model of incomplete information (version two) cont'd

Firm 1 knows exactly its marginal cost is high or low.

- If its marginal cost is **LOW**, i.e. $C_1(q_1) = c_L q_1$, then, given its belief on firm 2, it will solve

  $$\begin{aligned} &Max \quad \theta \times q_1[a-(q_1+q_2(c_H))-c_L]+(1-\theta)\times q_1[a-(q_1+q_2(c_L))-c_L] \\ &s.t. \quad\ q_1 \geq 0 \end{aligned}$$

- FOC:

  $$\theta[a-2q_1-q_2(c_H)-c_L]+(1-\theta)[a-2q_1-q_2(c_L)-c_L]=0$$

Hence, $q_1(c_L) = \dfrac{\theta[a-q_2(c_H)-c_L]+(1-\theta)[a-q_2(c_L)-c_L]}{2}$

- $q_1(c_L)$ is firm 1's best response to the belief that firm 2 chooses $q_2(c_H)$ with probability θ, and $q_2(c_L)$ with probability $1-\theta$, if firm 1's marginal cost is **LOW**.

Cournot duopoly model of incomplete information (version two) cont'd

- Firm 2 knows exactly its marginal cost is high or low before production.
- Firm 2 does not know exactly firm 1's marginal cost is high or low.
- But it believes that firm 1's cost function is
  - $C_1(q_1) = c_H q_1$ (in which it chooses $q_1(c_H)$) with probability θ.
  - $C_1(q_1) = c_L q_1$ (in which it chooses $q_1(c_L)$) with probability $1-\theta$.
- Now we solve firm 2's problem, given its belief on firm 1.

Cournot duopoly model of incomplete information (version two) cont'd

Firm 2 knows exactly its marginal cost is high or low.

- If its marginal cost is **HIGH**, i.e. $C_2(q_2) = c_H q_2$, then, given its belief on firm 1, it will solve

$$\begin{aligned} &Max \quad \theta \times q_2[a-(q_1(c_H)+q_2)-c_H] + (1-\theta)\times q_2[a-(q_1(c_L)+q_2)-c_H] \\ &s.t. \quad\; q_2 \geq 0 \end{aligned}$$

- FOC:

$$\theta[a - q_1(c_H) - 2q_2 - c_H] + (1-\theta)[a - q_1(c_L) - 2q_2 - c_H] = 0$$

Hence, $q_2(c_H) = \dfrac{\theta[a-q_1(c_H)-c_H]+(1-\theta)[a-q_1(c_L)-c_H]}{2}$

- $q_2(c_H)$ is firm 2's best response to the belief that firm 1 chooses $q_1(c_H)$ with probability θ, and $q_1(c_L)$ with probability $1-\theta$, if firm 2's marginal cost is **HIGH**.

Cournot duopoly model of incomplete information (version two) cont'd

Firm 2 knows exactly its marginal cost is high or low.

- If its marginal cost is **LOW**, i.e. $C_2(q_2) = c_L q_2$, then, given its belief on firm 1, it will solve

$$\begin{aligned} &Max \quad \theta \times q_2[a-(q_1(c_H)+q_2)-c_L] + (1-\theta)\times q_2[a-(q_1(c_L)+q_2)-c_L] \\ &s.t. \quad q_2 \ge 0 \end{aligned}$$

- FOC:

$$\theta[a-q_1(c_H)-2q_2-c_L]+(1-\theta)[a-q_1(c_L)-2q_2-c_L]=0$$

Hence, $q_2(c_L) = \dfrac{\theta[a-q_1(c_H)-c_L]+(1-\theta)[a-q_1(c_L)-c_L]}{2}$

- $q_2(c_L)$ is firm 2's best response to the belief that firm 1 chooses $q_1(c_H)$ with probability θ, and $q_1(c_L)$ with probability $1-\theta$, if firm 2's marginal cost is **LOW**.

Cournot duopoly model of incomplete information (version two) cont'd

- Now we have

$$q_1(c_H) = \frac{\theta[a - q_2(c_H) - c_H] + (1-\theta)[a - q_2(c_L) - c_H]}{2}$$

$$q_1(c_L) = \frac{\theta[a - q_2(c_H) - c_L] + (1-\theta)[a - q_2(c_L) - c_L]}{2}$$

$$q_2(c_H) = \frac{\theta[a - q_1(c_H) - c_H] + (1-\theta)[a - q_1(c_L) - c_H]}{2}$$

$$q_2(c_L) = \frac{\theta[a - q_1(c_H) - c_L] + (1-\theta)[a - q_1(c_L) - c_L]}{2}$$

- This is a symmetric model. So $q_1(c_H) = q_2(c_H)$ and $q_1(c_L) = q_2(c_L)$. Solving these four equations with four unknowns gives us.

$$q_1^*(c_H) = q_2^*(c_H) = \frac{1}{3}(a - c_H) - \frac{1-\theta}{6}(c_H - c_L)$$

$$q_1^*(c_L) = q_2^*(c_L) = \frac{1}{3}(a - c_L) + \frac{\theta}{6}(c_H - c_L)$$

Cournot duopoly model of incomplete information (version two) cont'd

- This can be written as $((q_1^*(c_H), q_1^*(c_L)), (q_2^*(c_H), q_2^*(c_L)))$

- If firm 1's marginal cost is HIGH then it chooses $q_1^*(c_H)$ which is its best response to firm 2's $(q_2^*(c_H), q_2^*(c_L))$ (and the probability).
- If firm 1's marginal cost is LOW then it chooses $q_1^*(c_L)$ which is its best response to firm 2's $(q_2^*(c_H), q_2^*(c_L))$ (and the probability).
- If firm 2's marginal cost is HIGH then it chooses $q_2^*(c_H)$ which is its best response to firm 1's $(q_1^*(c_H), q_1^*(c_L))$ (and the probability).
- If firm 2's marginal cost is LOW then it chooses $q_2^*(c_L)$ which is its best response to firm 1's $(q_1^*(c_H), q_1^*(c_L))$ (and the probability).

- **A Nash equilibrium solution called *Bayesian Nash equilibrium*.**

Battle of the sexes

- At the separate workplaces, Chris and Pat must choose to attend either an opera or a prize fight in the evening.
- Both Chris and Pat know the following:
  - Both would like to spend the evening together.
  - But Chris prefers the opera.
  - Pat prefers the prize fight.

		Pat: Opera	Pat: Prize Fight
Chris	Opera	2 , 1	0 , 0
	Prize Fight	0 , 0	1 , 2

Battle of the sexes with incomplete information (version one)

- Now Pat's preference depends on whether he is happy.
- If he is happy then his preference is the same.
- If he is unhappy then he prefers to spend the evening by himself and his preference is shown in the following table.
- Chris cannot figure out whether Pat is happy or not. But Chris believes that Pat is happy with probability 0.5 and unhappy with probability 0.5

Payoffs if Pat is unhappy

		Pat: Opera	Pat: Prize Fight
Chris	Opera	<u>2</u> , 0	0 , <u>2</u>
	Prize Fight	0 , <u>1</u>	<u>1</u> , 0

Battle of the sexes with incomplete information (version one) cont'd

- How to find a solution ?

Payoffs if Pat is happy with probability 0.5

Chris \ Pat	Opera	Prize Fight
Opera	2 , 1	0 , 0
Prize Fight	0 , 0	1 , 2

Payoffs if Pat is unhappy with probability 0.5

Chris \ Pat	Opera	Prize Fight
Opera	2 , 0	0 , 2
Prize Fight	0 , 1	1 , 0

Battle of the sexes with incomplete information (version one) cont'd

- Best response
  - If Chris chooses opera then Pat's best response: opera if he is happy, and prize fight if he is unhappy
  - Suppose that Pat chooses opera if he is happy, and prize fight if he is unhappy. What is Chris' best response?
    - If Chris chooses opera then she get a payoff 2 if Pat is happy, or 0 if Pat is unhappy. Her expected payoff is $2\times0.5+0\times0.5=1$
    - If Chris chooses prize fight then she get a payoff 0 if Pat is happy, or 1 if Pat is unhappy. Her expected payoff is $0\times0.5+1\times0.5=0.5$
    - Since $1>0.5$, Chris' best response is opera

  - A Bayesian Nash equilibrium: (opera, (opera if happy

Battle of the sexes with incomplete information (version one) cont'd

- Best response
  - If Chris chooses prize fight then Pat's best response: prize fight if he is happy, and opera if he is unhappy
  - Suppose that Pat chooses prize fight if he is happy, and opera if he is unhappy. What is Chris' best response?
    - If Chris chooses opera then she get a payoff 0 if Pat is happy, or 2 if Pat is unhappy. Her expected payoff is $0\times0.5+2\times0.5=1$
    - If Chris chooses prize fight then she get a payoff 1 if Pat is happy, or 0 if Pat is unhappy. Her expected payoff is $1\times0.5+0\times0.5=0.5$
    - Since $1>0.5$, Chris' best response is opera
  - (Prize, (prize fight if happy and opera if unhappy)) is

Cournot duopoly model of incomplete information (version three) cont'd

- Firm 2's cost depends on some factor (e.g. technology) that only firm 2 knows. Its cost can be
  - HIGH: cost function: $C_2(q_2)=c_H q_2$.
  - LOW: cost function: $C_2(q_2)=c_L q_2$.
- Firm 1's cost also depends on some other (independent or dependent) factor that only firm 1 knows. Its cost can be
  - HIGH: cost function: $C_1(q_1)=c_H q_1$.
  - LOW: cost function: $C_1(q_1)=c_L q_1$.

Cournot duopoly model of incomplete information (version three) cont'd

- Firm 1's quantity depends on its cost. It chooses
  - $q_1(c_H)$ if its cost is HIGH
  - $q_1(c_L)$ if its cost is LOW

- Firm 2's quantity also depends on its cost. It chooses
  - $q_2(c_H)$ if its cost is HIGH
  - $q_2(c_L)$ if its cost is LOW

Cournot duopoly model of incomplete information (version three) cont'd

- Before production, firm 1 knows exactly its cost is HIGH or LOW.
- However, firm 1 cannot know exactly firm 2's cost. Equivalently, it is uncertain about firm 2's payoff.
- Firm 1 believes that *if its cost is HIGH* then firm 2's cost function is
  - $C_2(q_2)=c_H q_2$ with probability $p_1(c_2=c_H \mid c_1=c_H)$, and
  - $C_2(q_2)=c_L q_2$ with probability $p_1(c_2=c_L \mid c_1=c_H)$.
- Firm 1 believes that *if its cost is LOW* then firm 2's cost function is
  - $C_2(q_2)=c_H q_2$ with probability $p_1(c_2=c_H \mid c_1=c_L)$, and
  - $C_2(q_2)=c_L q_2$ with probability $p_1(c_2=c_L \mid c_1=c_L)$.

- Example: $p_1(c_2=c_H \mid c_1=c_H)=p_1(c_2=c_H \mid c_1=c_L)=\theta$
  $p_1(c_2=c_L \mid c_1=c_H)=p_1(c_2=c_L \mid c_1=c_L)=1-\theta$ as in version two.

Cournot duopoly model of incomplete information (version three) cont'd

- Before production, firm 2 knows exactly its cost is HIGH or LOW.
- However, firm 2 cannot know exactly firm 1's cost. Equivalently, it is uncertain about firm 1's payoff.
- Firm 2 believes that *if its cost is HIGH* then firm 1's cost function is
  - $C_1(q_1) = c_H q_1$ with probability $p_2(c_1 = c_H \mid c_2 = c_H)$, and
  - $C_1(q_1) = c_L q_1$ with probability $p_2(c_1 = c_L \mid c_2 = c_H)$.
- Firm 2 believes that *if its cost is LOW* then firm 1's cost function is
  - $C_1(q_1) = c_H q_1$ with probability $p_2(c_1 = c_H \mid c_2 = c_L)$, and
  - $C_1(q_1) = c_L q_1$ with probability $p_2(c_1 = c_L \mid c_2 = c_L)$.
- Example: $p_2(c_1 = c_H \mid c_2 = c_H) = p_2(c_1 = c_H \mid c_2 = c_L) = \theta$
  $p_2(c_1 = c_L \mid c_2 = c_H) = p_2(c_1 = c_L \mid c_2 = c_L) = 1 - \theta$ as in version two.

Cournot duopoly model of incomplete information (version three) cont'd

Firm 1 knows exactly its cost is high or low.

- If its cost is **HIGH**, i.e. $C_1(q_1) = c_H q_1$, then, given its belief on firm 2, it will solve

$$
\begin{aligned}
Max \quad & p_1(c_2 = c_H \mid c_1 = c_H) \times \boxed{q_1[a - (q_1 + q_2(c_H)) - c_H]} \\
& + p_1(c_2 = c_L \mid c_1 = c_H) \times \boxed{q_1[a - (q_1 + q_2(c_L)) - c_H]} \\
s.t. \quad & q_1 \geq 0
\end{aligned}
$$

$\boxed{u_1(q_1, q_2(c_H); c_H)}$

$\boxed{u_1(q_1, q_2(c_L); c_H)}$

- FOC:

$$
\begin{aligned}
& p_1(c_2 = c_H \mid c_1 = c_H)[a - 2q_1 - q_2(c_H) - c_H] \\
& \quad + p_1(c_2 = c_L \mid c_1 = c_H)[a - 2q_1 - q_2(c_L) - c_H] = 0
\end{aligned}
$$

Hence,

$$
q_1(c_H) = \frac{a - c_H - p_1(c_2 = c_H \mid c_1 = c_H) q_2(c_H) - p_1(c_2 = c_L \mid c_1 = c_H) q_2(c_L)}{2}
$$

- $q_1(c_H)$ is firm 1's best response to its belief (probability) on firm 2's $(q_2(c_H),\ q_2(c_L))$ if firm 1's cost is **HIGH**.

Cournot duopoly model of incomplete information (version three) cont'd

Firm 1 knows exactly its cost is high or low.

- If its cost is **LOW**, i.e. $C_1(q_1) = c_L q_1$, then, given its belief on firm 2, it will solve

$$
\begin{aligned}
Max \quad & p_1(c_2 = c_H \mid c_1 = c_L) \times \boxed{q_1[a - (q_1 + q_2(c_H)) - c_L]} \\
& + p_1(c_2 = c_L \mid c_1 = c_L) \times \boxed{q_1[a - (q_1 + q_2(c_L)) - c_L]} \\
s.t. \quad & q_1 \ge 0
\end{aligned}
$$

$u_1(q_1, q_2(c_H); c_L)$

$u_1(q_1, q_2(c_L); c_L)$

- FOC:

$$
\begin{aligned}
& p_1(c_2 = c_H \mid c_1 = c_L)[a - 2q_1 - q_2(c_H) - c_L] \\
& \quad + p_1(c_2 = c_L \mid c_1 = c_L)[a - 2q_1 - q_2(c_L) - c_L] = 0
\end{aligned}
$$

Hence,

$$
q_1(c_L) = \frac{a - c_L - p_1(c_2 = c_H \mid c_1 = c_L) q_2(c_H) - p_1(c_2 = c_L \mid c_1 = c_L) q_2(c_L)}{2}
$$

- $q_1(c_L)$ is firm 1's best response to its belief (probability) on firm 2's $(q_2(c_H),\ q_2(c_L))$ if firm 1's cost is **LOW**.

Cournot duopoly model of incomplete information (version three) cont'd

Firm 2 knows exactly its cost is high or low.

- If its cost is **HIGH**, i.e. $C_2(q_2) = c_H q_2$, then, given its belief on firm 1, it will solve

$$\begin{aligned} Max \quad & p_2(c_1 = c_H \mid c_2 = c_H) \times \underbrace{\boxed{q_2[a-(q_1(c_H)+q_2)-c_H]}}_{u_2(q_1(c_H),\, q_2;\, c_H)} \\ & + p_2(c_1 = c_L \mid c_2 = c_H) \times \underbrace{\boxed{q_2[a-(q_1(c_L)+q_2)-c_H]}}_{u_2(q_1(c_L),\, q_2;\, c_H)} \\ s.t. \quad & q_2 \geq 0 \end{aligned}$$

- FOC:

$$\begin{aligned} & p_2(c_1 = c_H \mid c_2 = c_H)[a - q_1(c_H) - 2q_2 - c_H] \\ & + p_2(c_1 = c_L \mid c_2 = c_H)[a - q_1(c_L) - 2q_2 - c_H] = 0 \end{aligned}$$

Hence,

$$q_2(c_H) = \frac{a - c_H - p_2(c_1 = c_H \mid c_2 = c_H) q_1(c_H) - p_2(c_1 = c_L \mid c_2 = c_H) q_1(c_L)}{2}$$

- $q_2(c_H)$ is firm 2's best response to its belief (probability) on firm 1's $(q_1(c_H), q_1(c_L))$ if firm 2's cost is **HIGH**.

Cournot duopoly model of incomplete information (version three) cont'd

Firm 2 knows exactly its cost is high or low.

- If its cost is **LOW**, i.e. $C_2(q_2) = c_L q_2$, then, given its belief on firm 1, it will solve

$$
\begin{aligned}
Max \quad & p_2(c_1 = c_H \mid c_2 = c_L) \times \underbrace{\boxed{q_2[a - (q_1(c_H) + q_2) - c_L]}}_{u_2(q_1(c_H),\, q_2;\, c_L)} \\
& + p_2(c_1 = c_L \mid c_2 = c_L) \times \underbrace{\boxed{q_2[a - (q_1(c_L) + q_2) - c_L]}}_{u_2(q_1(c_L),\, q_2;\, c_L)} \\
s.t. \quad & q_2 \geq 0
\end{aligned}
$$

- FOC:

$$
\begin{aligned}
& p_2(c_1 = c_H \mid c_2 = c_L)[a - q_1(c_H) - 2q_2 - c_L] \\
& + p_2(c_1 = c_L \mid c_2 = c_L)[a - q_1(c_L) - 2q_2 - c_L] = 0
\end{aligned}
$$

Hence,

$$
q_2(c_L) = \frac{a - c_L - p_2(c_1 = c_H \mid c_2 = c_L) q_1(c_H) - p_2(c_1 = c_L \mid c_2 = c_L) q_1(c_L)}{2}
$$

- $q_2(c_L)$ is firm 2's best response to its belief (probability) on firm 1's $(q_1(c_H), q_1(c_L))$ if firm 2's cost is **LOW**.

Cournot duopoly model of incomplete information (version three) cont'd

- Now we have four equations with four unknowns.

$$q_1(c_H) = \frac{a - c_H - p_1(c_2 = c_H \mid c_1 = c_H) q_2(c_H) - p_1(c_2 = c_L \mid c_1 = c_H) q_2(c_L)}{2}$$

$$q_1(c_L) = \frac{a - c_L - p_1(c_2 = c_H \mid c_1 = c_L) q_2(c_H) - p_1(c_2 = c_L \mid c_1 = c_L) q_2(c_L)}{2}$$

$$q_2(c_H) = \frac{a - c_H - p_2(c_1 = c_H \mid c_2 = c_H) q_1(c_H) - p_2(c_1 = c_L \mid c_2 = c_H) q_1(c_L)}{2}$$

$$q_2(c_L) = \frac{a - c_L - p_2(c_1 = c_H \mid c_2 = c_L) q_1(c_H) - p_2(c_1 = c_L \mid c_2 = c_L) q_1(c_L)}{2}$$

- Solving these gives us the following Bayesian Nash equilibrium.

$$\left(q_1^*(c_H),\ q_1^*(c_L)\right)$$
$$\left(q_2^*(c_H),\ q_2^*(c_L)\right)$$

Cournot duopoly model of incomplete information (version three) cont'd

- The Bayesian Nash equilibrium: $((q_1^*(c_H),\ q_1^*(c_L)),\ (q_2^*(c_H),\ q_2^*(c_L)))$

- If firm 1's marginal cost is HIGH then it chooses $q_1^*(c_H)$ which is its best response to firm 2's $(q_2^*(c_H),\ q_2^*(c_L))$ (and the probability).
- If firm 1's marginal cost is LOW then it chooses $q_1^*(c_L)$ which is its best response to firm 2's $(q_2^*(c_H),\ q_2^*(c_L))$ (and the probability).
- If firm 2's marginal cost is HIGH then it chooses $q_2^*(c_H)$ which is its best response to firm 1's $(q_1^*(c_H),\ q_1^*(c_L))$ (and the probability).
- If firm 2's marginal cost is LOW then it chooses $q_2^*(c_L)$ which is its best response to firm 1's $(q_1^*(c_H),\ q_1^*(c_L))$ (and the probability).

Normal-form representation of static Bayesian games

- The normal-form representation of an *n*-player static game *G* of incomplete information specifies:
  - A finite set of players {1, 2, ..., *n*},
  - players' action sets $A_1,\ A_2,\ A_3,\ ...,A_n$ and
  - their payoff functions
  - more

- Remark: a player's payoff function depends on not only the *n* players' actions but also her TYPE.
- T_i is player *i* 's type set.
- Example: $T_1 = \{c_H,\ c_L\}$, $T_2 = \{c_H,\ c_L\}$

Normal-form representation of static Bayesian games: payoffs

- Player *i*'s payoff function is represented as:
  $u_i(a_1, a_2, ..., a_n; t_i)$ for $a_1 \in A_1, a_2 \in A_2, ..., a_n \in A_n, t_i \in T_i$.
- Example: $u_1(q_1, q_2; c_H) = q_1[a - (q_1 + q_2) - c_H]$
  $u_1(q_1, q_2; c_L) = q_1[a - (q_1 + q_2) - c_L]$

- Each player knows her own type. Equivalently, she knows her own payoff function.
- Each player may be uncertain about other players' types. Equivalently, she is uncertain about other players' payoff functions.

Normal-form representation of static Bayesian games: beliefs (probabilities)

- Player *i* has beliefs on other players' types, denoted by

$$p_i(t_1, t_2, ..., t_{i-1}, t_{i+1}, ..., t_n \mid t_i) \text{ for } t_1 \in T_1, t_2 \in T_2, ..., t_n \in T_n. \text{ or}$$

$$p_i(t_{-i} \mid t_i) \text{ where } t_{-i} = (t_1, t_2, ..., t_{i-1}, t_{i+1}, ..., t_n),\ t_1 \in T_1, t_2 \in T_2, ..., t_n \in T_n.$$

- Player *i*'s beliefs are conditional probabilities

- Example:

$$p_1(c_2 = c_H \mid c_1 = c_H)$$
$$p_1(c_2 = c_H \mid c_1 = c_L)$$
$$p_1(c_2 = c_L \mid c_1 = c_H)$$
$$p_1(c_2 = c_L \mid c_1 = c_L)$$

Strategy

- In a static Bayesian game, a strategy for player *i* is a function $s_i(t_i)$ for each $t_i \in T_i$.

- $s_i(t_i)$ specifies what player *i* does for her each type $t_i \in T_i$

- Example: $(q_1(c_H), q_1(c_L))$ is a strategy for firm 1 in the Cournot model of incomplete information (version three).

Bayesian Nash equilibrium: 2-player

- In a static Bayesian 2-player game $\{A_1, A_2; T_1, T_2; p_1, p_2; u_1, u_2\}$, the strategies $s_1^*(\bullet), s_2^*(\bullet)$ are pure strategy Bayesian Nash equilibrium if
  - for each of player 1's types $t_1 \in T_1$, $s_1^*(t_1)$ solves

$$\underset{a_1 \in A_1}{Max} \sum_{t_2 \in T_2} u_1(a_1, s_2^*(t_2); t_1) p_1(t_2 \mid t_1)$$

  - and for each of player 2's types $t_2 \in T_2$, $s_2^*(t_2)$ solves

$$\underset{a_2 \in A_2}{Max} \sum_{t_1 \in T_1} u_2(s_1^*(t_1), a_2; t_2) p_2(t_1 \mid t_2)$$

Bayesian Nash equilibrium: 2-player

- In a static Bayesian 2-player game $\{A_1, A_2; T_1, T_2; p_1, p_2; u_1, u_2\}$, the strategies $s_1^*(\bullet), s_2^*(\bullet)$ are pure strategy Bayesian Nash equilibrium if for **each *i* and *j*,** (assume $T_1 = \{t_{11}, t_{12}, ...\}, T_2 = \{t_{21}, t_{22}, ...\}$)

$s_1^*(t_{11})$

$s_1^*(t_{12})$

$\vdots$

$\vdots$

$\boxed{s_1^*(t_{1i})}$

$\vdots$

$s_1^*(t_{1n})$

$\vdots$

In the sense of expectation based on her belief

player 2's best response if her type is t_{2j}

$s_2^*(t_{21})$

$s_2^*(t_{22})$

$\vdots$

$\boxed{s_2^*(t_{2j})}$

$\vdots$

$s_2^*(t_{2n})$

$\vdots$

player 1's best response if her type is t_{1i}

In the sense of expectation based on her belief

Static Bayesian games

- The normal-form representation of an *n*-player static game *G* of incomplete information specifies:
  - A finite set of players {1, 2, ..., *n*},
  - players' action sets $A_1,\ A_2,\ A_3,\ ...,A_n,$
  - players' type sets $T_1,\ T_2,\ T_3,\ ...,T_n,$
  - players' beliefs $p_1,\ p_2,\ p_3,\ ...,\ p_n$
  - their payoff functions
    $u_i(a_1, a_2, ..., a_n; t_i)$ for $a_1 \in A_1, a_2 \in A_2, ..., a_n \in A_n, t_i \in T_i.$

- In a static Bayesian game, a strategy for player *i* is a function $s_i(t_i)$ for each $t_i \in T_i$.

- $s_i(t_i)$ specifies what player *i* does for her each type $t_i \in T_i$

Bayesian Nash equilibrium: 2-player

- In a static Bayesian 2-player game $\{A_1, A_2; T_1, T_2; p_1, p_2; u_1, u_2\}$, the strategies $s_1^*(\bullet), s_2^*(\bullet)$ are pure strategy Bayesian Nash equilibrium if
  - for each of player 1's types $t_1 \in T_1$, $s_1^*(t_1)$ solves

$$\underset{a_1 \in A_1}{Max} \sum_{t_2 \in T_2} u_1(a_1, s_2^*(t_2); t_1) p_1(t_2 \mid t_1)$$

  - and for each of player 2's types $t_2 \in T_2$, $s_2^*(t_2)$ solves

$$\underset{a_2 \in A_2}{Max} \sum_{t_1 \in T_1} u_2(s_1^*(t_1), a_2; t_2) p_2(t_1 \mid t_2)$$

Bayesian Nash equilibrium: 2-player

- In a static Bayesian 2-player game $\{A_1, A_2; T_1, T_2; p_1, p_2; u_1, u_2\}$, the strategies $s_1^*(\bullet), s_2^*(\bullet)$ are pure strategy Bayesian Nash equilibrium if for **each *i* and *j*,** (assume $T_1 = \{t_{11}, t_{12}, \ldots.\}, T_2 = \{t_{21}, t_{22}, \ldots.\}$)

$s_1^*(t_{11})$
$s_1^*(t_{12})$
$\vdots$
$\vdots$
$\boxed{s_1^*(t_{1i})}$
$\vdots$
$s_1^*(t_{1n})$
$\vdots$

In the sense of expectation based on her belief

player 2's best response if her type is $\boldsymbol{t_{2j}}$

player 1's best response if her type is $\boldsymbol{t_{1i}}$

In the sense of expectation based on her belief

$s_2^*(t_{21})$
$s_2^*(t_{22})$
$\vdots$
$\boxed{s_2^*(t_{2j})}$
$\vdots$
$s_2^*(t_{2n})$
$\vdots$

Battle of the sexes

- At the separate workplaces, Chris and Pat must choose to attend either an opera or a prize fight in the evening.
- Both Chris and Pat know the following:
  - Both would like to spend the evening together.
  - But Chris prefers the opera.
  - Pat prefers the prize fight.

		Pat: Opera	Pat: Prize Fight
Chris	Opera	2 , 1	0 , 0
	Prize Fight	0 , 0	1 , 2

Battle of the sexes with incomplete information (version two)

- Pat’s preference depends on whether he is happy. If he is happy then his preference is the same.
- If he is unhappy then he prefers to spend the evening by himself.
- Chris cannot figure out whether Pat is happy or not. But Chris believes that Pat is happy with probability 0.5 and unhappy with probability 0.5
- Chris’ preference also depends on whether she is happy. If she is happy then her preference is the same.
- If she is unhappy then she prefers to spend the evening by herself.
- Pat cannot figure out whether Chris is happy or not. But Pat believes that Chris is happy with probability 2/3 and unhappy with probability 1/3.

Battle of the sexes with incomplete information (version two) cont'd

Chris is happy
Pat is happy

Chris \ Pat	Opera	Fight
Opera	2 , 1	0 , 0
Fight	0 , 0	1 , 2

Chris is happy
Pat is unhappy

Chris \ Pat	Opera	Fight
Opera	2 , 0	0 , 2
Fight	0 , 1	1 , 0

Chris is unhappy
Pat is happy

Chris \ Pat	Opera	Fight
Opera	0 , 1	2 , 0
Fight	1 , 0	0 , 2

Chris is unhappy
Pat is unhappy

Chris \ Pat	Opera	Fight
Opera	0 , 0	2 , 2
Fight	1 , 1	0 , 0

- Check whether ((Opera if happy, Opera if unhappy), (Opera if happy, Fight is unhappy)) is a Bayesian NE

Battle of the sexes with incomplete information (version two) cont'd

- Check whether ((Opera if happy, Opera if unhappy), (Opera if happy, Fight is unhappy)) is a Bayesian Nash equilibrium.
  - Chris' best response to Pat's (Opera if happy, Fight is unhappy) <u>if Chris is HAPPY</u>
    - If Chris chooses Opera then she gets a payoff 2 if Pat is happy (probability 0.5), or a payoff 0 if Pat is unhappy (probability 0.5). Her expected payoff=$2\times0.5+0\times0.5=1$
    - If Chris chooses Fight then she gets a payoff 0 if Pat is happy (probability 0.5), or a payoff 1 if Pat is unhappy (probability 0.5). Her expected payoff=$0\times0.5+1\times0.5=0.5$
    - Hence, Chris' best response is Opera if she is HAPPY.

Battle of the sexes with incomplete information (version two) cont'd

- Check whether ((Opera if happy, Opera if unhappy), (Opera if happy, Fight is unhappy)) is a Bayesian Nash equilibrium.
  - Chris' best response to Pat's (Opera if happy, Fight is unhappy) <u>if Chris is UNHAPPY</u>
    - If Chris chooses Opera then she gets a payoff 0 if Pat is happy (probability 0.5), or a payoff 2 if Pat is unhappy (probability 0.5). Her expected payoff=$0\times0.5+2\times0.5=1$
    - If Chris chooses Fight then she gets a payoff 1 if Pat is happy (probability 0.5), or a payoff 0 if Pat is unhappy (probability 0.5). Her expected payoff=$1\times0.5+0\times0.5=0.5$
    - Hence, Chris' best response is Opera if she is UNHAPPY.

Battle of the sexes with incomplete information (version two) cont'd

- Check whether ((Opera if happy, Opera if unhappy), (Opera if happy, Fight is unhappy)) is a Bayesian Nash equilibrium.
  - Pat's best response to Chris' (Opera if happy, Opera if unhappy) <u>if Pat is HAPPY</u>
    - If Pat chooses Opera then he gets a payoff 1 if Chris is happy (probability 2/3), or a payoff 1 if Chris is unhappy (probability 1/3). His expected payoff=$1\times(2/3)+1\times(1/3)=1$
    - If Pat chooses Fight then he gets a payoff 0 if Chris is happy (probability 2/3), or a payoff 0 if Chris is unhappy (probability 1/3). His expected payoff=$0\times(2/3)+0\times(1/3)=0$
    - Hence, Pat's best response is Opera if he is HAPPY.

Battle of the sexes with incomplete information (version two) cont'd

- Check whether ((Opera if happy, Opera if unhappy), (Opera if happy Fight is unhappy)) is a Bayesian Nash equilibrium.
  - Pat's best response to Chris' (Opera if happy, Opera if unhappy) if Pat is UNHAPPY
    - If Pat chooses Opera then he gets a payoff 0 if Chris is happy (probability 2/3), or a payoff 0 if Chris is unhappy (probability 1/3). His expected payoff=$0\times(2/3)+1\times(1/3)=0$
    - If Pat chooses Fight then he gets a payoff 2 if Chris is happy (probability 2/3), or a payoff 2 if Chris is unhappy (probability 1/3). His expected payoff=$2\times(2/3)+2\times(1/3)=2$
    - Hence, Pat's best response is Fight if he is UNHAPPY.

- Hence, ((Opera if happy, Opera if unhappy), (Opera if happy, Fight is unhappy)) is a Bayesian Nash equilibrium.

Battle of the sexes with incomplete information (version two) cont'd

Chris believes that Pat is happy with probability 0.5, unhappy 0.5

Chris is happy

		Pat (0.5, 0.5)			
		(O,O)	**(O,F)**	**(F,O)**	**(F,F)**
Chris	O	2	1	1	0
	F	0	1/2	1/2	1

Chris is unhappy

		Pat (0.5, 0.5)			
		(O,O)	**(O,F)**	**(F,O)**	**(F,F)**
Chris	O	0	1	1	2
	F	1	1/2	1/2	0

Chris' expected payoff of playing Fight if Chris is happy and Pat plays (Opera if happy, Fight if unhappy)

Battle of the sexes with incomplete information (version two) cont'd

Pat believes that Chris is happy with probability 2/3, unhappy 1/3

Pat is happy

Chris (2/3, 1/3)

	Pat: O	Pat: F
(O,O)	1	0
(O,F)	2/3	2/3
(F,O)	1/3	4/3
(F,F)	0	2

Pat is unhappy

Chris (2/3, 1/3)

	Pat: O	Pat: F
(O,O)	0	2
(O,F)	1/3	4/3
(F,O)	2/3	2/3
(F,F)	1	0

Pat's expected payoff of playing Opera if Pat is unhappy and Chris plays (Fight if happy, Fight if unhappy)

Battle of the sexes with incomplete information (version two) cont'd

- Check whether ((Fight if happy, Opera if unhappy), (Fight if happy, Fight is unhappy)) is a Bayesian Nash equilibrium.

First-price sealed-bid auction (3.2.B of Gibbons)

- A single good is for sale.
- Two bidders, 1 and 2, simultaneously submit their bids.
- Let b_1 denote bidder 1's bid and b_2 denote bidder 2's bid
- The higher bidder wins the good and pays the price she bids
- The other bidder gets and pays nothing
- In case of a tie, the winner is determined by a flip of a coin
- Bidder *i* has a valuation $v_i \in [0, 1]$ for the good. v_1 and v_2 are independent.
- Bidder 1 and 2's payoff functions:

$$u_1(b_1, b_2; v_1) = \begin{cases} v_1 - b_1 & \text{if } b_1 > b_2 \\ \dfrac{v_1 - b_1}{2} & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases} \qquad u_2(b_1, b_2; v_2) = \begin{cases} v_2 - b_2 & \text{if } b_2 > b_1 \\ \dfrac{v_2 - b_2}{2} & \text{if } b_2 = b_1 \\ 0 & \text{if } b_2 < b_1 \end{cases}$$

First-price sealed-bid auction (3.2.B of Gibbons) cont'd

- Normal form representation:
  - Two bidders, 1 and 2
  - Action sets (bid sets): $A_1 \in [0,\ \infty),\ A_2 \in [0,\ \infty)$
  - Type sets (valuations sets): $T_1 \in [0,\ 1],\ T_2 \in [0,\ 1]$
  - Beliefs:
    Bidder 1 believes that v_2 is uniformly distributed on $[0,\ 1]$.
    Bidder 2 believes that v_1 is uniformly distributed on $[0,\ 1]$.
    v_1 and v_2 are independent.
  - Bidder 1 and 2's payoff functions:

$$u_1(b_1, b_2; v_1) = \begin{cases} v_1 - b_1 & \text{if } b_1 > b_2 \\ \dfrac{v_1 - b_1}{2} & \text{if } b_1 = b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases} \qquad u_2(b_1, b_2; v_2) = \begin{cases} v_2 - b_2 & \text{if } b_2 > b_1 \\ \dfrac{v_2 - b_2}{2} & \text{if } b_2 = b_1 \\ 0 & \text{if } b_2 < b_1 \end{cases}$$

First-price sealed-bid auction (3.2.B of Gibbons) cont'd

- A strategy for bidder 1 is a function $b_1(v_1)$, for all $v_1 \in [0,1]$.
- A strategy for bidder 2 is a function $b_2(v_2)$, for all $v_2 \in [0,1]$.

- Given bidder 1's belief on bidder 2, for each $v_1 \in [0,1]$, bidder 1 solves

$$\underset{b_1 \geq 0}{Max} \quad (v_1 - b_1)\mathrm{Prob}\{b_1 > b_2(v_2)\} + \frac{1}{2}(v_1 - b_1)\mathrm{Prob}\{b_1 = b_2(v_2)\}$$

- Given bidder 2's belief on bidder 1, for each $v_2 \in [0,1]$, bidder 2 solves

$$\underset{b_2 \geq 0}{Max} \quad (v_2 - b_2)\mathrm{Prob}\{b_2 > b_1(v_1)\} + \frac{1}{2}(v_2 - b_2)\mathrm{Prob}\{b_2 = b_1(v_1)\}$$

First-price sealed-bid auction (3.2.B of Gibbons) cont'd

- Check whether $\left(b_1^*(v_1) = \frac{v_1}{2},\ b_2^*(v_2) = \frac{v_2}{2} \right)$ is Bayesian Nash equilibrium.
- Given bidder 1's belief on bidder 2, for each $v_1 \in [0,1]$, bidder 1's best response to $b_2^*(v_2)$ solves

$$\underset{b_1 \geq 0}{Max} \quad (v_1 - b_1)\text{Prob}\{b_1 > b_2^*(v_2)\} + \frac{1}{2}(v_1 - b_1)\text{Prob}\{b_1 = b_2^*(v_2)\}$$

$$\underset{b_1 \geq 0}{Max} \quad (v_1 - b_1)\text{Prob}\{b_1 > \frac{v_2}{2}\} + \frac{1}{2}(v_1 - b_1)\text{Prob}\{b_1 = \frac{v_2}{2}\}$$

$$\underset{b_1 \geq 0}{Max} \quad (v_1 - b_1)\text{Prob}\{v_2 < 2b_1\} + \frac{1}{2}(v_1 - b_1)\text{Prob}\{v_2 = 2b_1\}$$

$$\underset{b_1 \geq 0}{Max} \quad (v_1 - b_1)2b_1$$

FOC: $$2v_1 - 4b_1 = 0 \quad \Rightarrow \quad b_1(v_1) = \frac{v_1}{2}$$

First-price sealed-bid auction (3.2.B of Gibbons) cont'd

- Hence, for each $v_1 \in [0,1]$, $b_1^*(v_1) = \frac{v_1}{2}$ is bidder 1's best response to bidder 2's $b_2^*(v_2) = \frac{v_2}{2}$.
- By symmetry, for each $v_2 \in [0,1]$, $b_2^*(v_2) = \frac{v_2}{2}$ is bidder 2's best response to bidder 1's $b_1^*(v_1) = \frac{v_1}{2}$.
- Therefore, $\left(b_1^*(v_1) = \frac{v_1}{2},\ b_2^*(v_2) = \frac{v_2}{2}\right)$ is Bayesian Nash equilibrium.
- Show that in a second-price sealed-bid auction, it is a NE for everyone to truthfully reveal their preferences.

First-price sealed-bid auction (3.2.B of Gibbons) cont'd

The Basic **Trade-off** facing a bidder:

The higher the bid, the more likely the bidder is to win; the lower the bid, the larger the gain if the bidder does win.